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Small x behavior of parton distributions. A study of higher twist effects.

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Higher twist corrections to F_2 at small x are studied for the case of a flat initial condition for the twist-two QCD evolution in the next-to-leading order approximation. We present an analytical parameterization of the contributions from the twist-two and higher twist operators of the Wilson operator product expansion. Higher twist terms are evaluated using two different approaches, one motivated by BFKL and the other motivated by the renormalon formalism. The results of the latter approach are in very good agreement with deep inelastic scattering data from HERA.

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I. INTRODUCTION

For more than a decade various models on the behavior of quarks and gluons at small x has been confronted with a large amount of experimental data from HERA on the deep-inelastic scattering (DIS) structure function F_2 [1–14]. In the small x regime, non-perturbative effects are expected to give a substantial contribution to F_2. However, what is observed up to very low $Q^2 \sim 1$ GeV^2 values, traditionally explained by soft processes, is described reasonably well by perturbative QCD evolution (see for example [15]). Thus, it is important to identify the kinematical region where the well-established perturbative QCD formalism can be safely applied.

At small x the Q^2 dependence of quarks and gluons is usually obtained from the numerical solution of the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) equations [16–20] [129]. The x profile of partons at some initial Q_0^2 and the QCD energy scale Λ are determined from a fit to experimental data [21–34].

On the other hand, when analyzing exclusively the small x region, a much simpler analysis can be done by using some of the existing analytical approaches of DGLAP equations in the small x limit [35–44]. In Refs. [35–37, 43, 44] it was pointed out that HERA small x data can be interpreted in terms of the so called doubled asymptotic scaling (DAS) phenomenon related to the asymptotic behavior of the DGLAP evolution discovered many years ago in [16, 17, 45].

In the present work we incorporate the contribution from higher twist (HT) operators of the Wilson operator product expansion to our previous analysis [44]. The semi-analytical solution of DGLAP equations obtained in Ref. [44] using a flat initial condition, is the next-to-leading order (NLO) extension of previous studies performed at the leading order (LO) in perturbative QCD [35, 43]. The flat initial conditions at some initial value Q_0^2 correspond to the case of parton distributions tending to some constant when $x \to 0$.

In Ref. [44], both the gluon and quark singlet densities are presented in terms of the diagonal $'+'$ and $'-'$ components obtained from the DGLAP equations in the Mellin moment space. The $'-'$ components are constants at small x for any values of Q^2, whereas the $'+'$ components grow for $Q^2 \geq Q_0^2$ as [130]

$$\sim \exp\left(2\sqrt{\left[a_+ \ln\left(\frac{a_s(Q_0^2)}{a_s(Q^2)}\right) - \left(b_+ + a_+\frac{\beta_1}{\beta_0}\right)\left(a_s(Q_0^2) - a_s(Q^2)\right)\right]\ln\left(\frac{1}{z}\right)}\right), \tag{1}$$

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where $a_+ = 4C_A/\beta_0$ and $b_+ = 8[23C_A - 26C_F]T_R f/(9\beta_0)$. In Eq. (1) and hereafter we use the notation $a_s = \alpha_s/(4\pi)$.

The first two coefficients of the QCD β-function in the $\overline{\rm MS}$-scheme are $\beta_0 = (11/3)C_A - (4/3)T_R f$ and $\beta_1 = (2/3)[17C_A^2 - 10C_A T_R f - 6C_F T_R f]$ where f is the number of active flavors. This new presentation as a function of the $SU(N)$ group Casimirs, with f active flavors, $C_A = N$, $T_R = 1/2$, $T_F = T_R f$ and $C_F = (N^2-1)/(2N)$ permits one to apply our results to, for example, the popular $N=1$ supersymmetric model. Of course, for $N=3$ one obtains the QCD result [44].

The analysis performed in our previous work [44] has shown very good agreement with H1 and ZEUS 1994 data [4, 11, 12] at $Q^2 \geq 1.5$ GeV^2. Here, we add the contribution from higher twist operators with the hope to describe also more modern data [2, 3, 7–10] at lower Q^2

Moreover, in comparison with Ref. [44], in the present work we have solved the technical problem of "backward" evolution that leads us now to have the normalization scale Q_0^2 of DGLAP evolution in the middle point of the Q^2 range.

A. Basic formulae

At this point of the introduction, we find convenient to present the basic results of our article: the twist-four and twist-six corrections to F_2 in the DAS approach. Thus, a reader who has interest only in application of the formulas to the analysis of F_2 can skip the following sections and start to read Section X, where the fits of F_2 are performed. We note, however, that some of the sections that follows contain also the contribution of power corrections to the derivatives $\partial F_2/\partial \ln Q^2$ and $\partial \ln F_2/\partial \ln(1/x)$ and to the parton distributions (PD) (see the Sections VI, VII, VIII and IX, respectively).

The basic results of the present article are the twist-four and twist-six corrections to F_2

$$F_2(x,Q^2) \; = \; F_2^{\tau 2}(x,Q^2) \; + \; \frac{1}{Q^2}\, F_2^{\tau 4}(z,Q^2) \; + \; \frac{1}{Q^4}\, F_2^{\tau 6}(z,Q^2) \; , \tag{2}$$

where for the higher twist parts $F_2^{\tau 4,6}$ BFKL-motivated evaluations [46–49] (in this case only the twist-four correction has been estimated) and the calculations [50] in the framework of the renormalon model (hereafter marked with superindex R) have been used.

The latter case is essentially more complete and the predicted HT corrections can be expressed through the twist-two ones as follows

$$F_2^{R\tau 4}(z,Q^2) \; = \; e \sum_{a=q,G} \mathrm{a}_a^{\tau 4}\, \widetilde{\mu}_a^{\tau 4}(z,Q^2) \otimes f_a^{\tau 2}(z,Q^2) \; = \; \sum_{a=q,G} F_{2,a}^{R\tau 4}(z,Q^2) \; , \tag{3}$$

where the symbol $\otimes$ marks the Mellin convolution (see Eq. (55) below), the functions $\widetilde{\mu}_a^{\tau 4}(z,Q^2)$ are given in [50] and $e = (\sum_1^f e_i^2)/f$ is the average charge square for f active quarks. We call $F_{2,q}^{R\tau 4}$ and $F_{2,G}^{R\tau 4}$, the HT corrections proportional to the twist-two quark and gluon densities, respectively.

Note that the parton distributions $f_a^{\tau 2}(z,Q^2)$ are multiplied by z, i.e., $f_q^{\tau 2}(z,Q^2) = z\, q(z,Q^2)$ and $f_G^{\tau 2}(z,Q^2) = z\, G(z,Q^2)$. Note also that we neglect the non-singlet quark density $f_\Delta(z,Q^2)$ and the valent part $f_V(z,Q^2)$ of the singlet quark distributions, because they have the following small-x asymptotics: $f_\Delta(z,Q^2) \sim f_V(z,Q^2) \sim x^{\lambda_V}$, where $\lambda_V \sim 0.3 \div 0.5$. Thus, our quark density $f_a^{\tau 2}(z,Q^2)$ contains only the sea part $f_S(z,Q^2)$, i.e. $f_a^{\tau 2}(z,Q^2) = f_S(z,Q^2)$.

For the leading twist part we have [44] at the LO and NLO approximations, respectively,

$$F_{2,\mathrm{LO}}^{\tau 2}(z,Q^2) \; = \; e\, f_{q,\mathrm{LO}}^{\tau 2}(z,Q^2) \; , \tag{4a}$$

$$F_2^{\tau 2}(z,Q^2) \; = \; e \left(f_q^{\tau 2}(z,Q^2) \; + \; \frac{4T_R f}{3}\, a_s(Q^2)\, f_G^{\tau 2}(z,Q^2) \right) \; . \tag{4b}$$

Let us keep the NLO relation (4b) beyond the leading twist approximation. Then for the total F_2 (see Eq. (2)) we obtain

$$F_2(z,Q^2) \; = \; e \left(f_q(z,Q^2) \; + \; \frac{4T_R f}{3}\, a_s(Q^2)\, f_G(z,Q^2) \right) \; , \tag{5}$$

where $f_a(z,Q^2)$ are the parton distributions containing both the twist-two part [44] (see next Section) and the twist-four and twist-six contributions

$$f_a(x,Q^2) \; = \; f_a^{\tau 2}(x,Q^2) \; + \; \frac{1}{Q^2}\, f_a^{R\tau 4}(z,Q^2) \; + \; \frac{1}{Q^4}\, f_a^{R\tau 6}(z,Q^2) \; . \tag{6}$$

For the HT part $f_a^{R\tau 4,6}(z,Q^2)$ calculations in the framework of the renormalon model have been used [131].

We would like to note that each HT term $f_a^{R\tau 4,6}(z,Q^2)$ can be chosen in a quite arbitrary form and only the combination

$$f_q^{R\tau 4,6}(z,Q^2) \; + \; \frac{4T_Rf}{3}\, a_s(Q^2)\, f_G^{R\tau 4,6}(z,Q^2) \tag{7}$$

is unique, because we kept the original twist-two relation, Eq. (4b), to be same when HT corrections are incorporated (see Eq. (5)).

Note that in our previous studies [51–53] we did not use the Eq. (5) to parameterize the HT corrections to F_2. Instead we consider the following representation

$$F_2^{R\tau 4,6}(z,Q^2) \;=\; e\, \hat{f}_q^{R\tau 4,6}(z,Q^2) \;, \tag{8}$$

coming from the LO relation (4a) between F_2 and parton distributions. The choice (8) looks quite natural because HT corrections have been obtained in [50] at the LO approximation. However, this choice is only useful to fit F_2 data and it has no interest to study the parton distributions themselves: note that the HT corrections to the gluon density are absent in Eq. (8). Indeed, in the calculation of F_2 at NLO one has to take a gluon density as in the r.h.s. of the Eq. (4b). So, one should take the condition

$$\hat{f}_G^{R\tau 4,6}(z,Q^2) \;=\; 0 \;, \tag{9}$$

which is not so natural. Moreover, the choice (8) and (9) leads to a quite complicated form for the HT corrections to the quark density: there are two independent contributions $\sim A_q^{\tau 2}$ and $\sim A_G^{\tau 2}$ (see Refs. [51–53] and formulas therein).

In the work we also study x and Q^2 dependences of $\partial F_2/\partial \ln Q^2$ and $\partial \ln F_2/\partial \ln(1/x)$, that force to define the parton densities in a proper way. So, we take another quite *natural* choice

$$f_q^{R\tau 4,6}(z,Q^2) \;=\; \mathrm{a}_q^{\tau 4,6}\, \widetilde{\mu}_q^{\tau 4,6}(z,Q^2) \otimes f_q^{\tau 2}(z,Q^2) \;\equiv\; \frac{1}{e}\, F_{2,q}^{R\tau 4,6}(z,Q^2) \;, \tag{10a}$$

$$f_G^{R\tau 4,6}(z,Q^2) \;=\; \frac{3/4T_Rf}{a_s(Q^2)}\, \mathrm{a}_G^{\tau 4,6}\, \widetilde{\mu}_G^{\tau 4,6}(z,Q^2) \otimes f_G^{\tau 2}(z,Q^2) \;\equiv\; \frac{3/4T_Rf}{e a_s(Q^2)}\, F_{2,G}^{R\tau 4,6}(z,Q^2) \;, \tag{10b}$$

i.e., the HT quark (gluon) part of F_2 relates only to the corresponding quark (gluon) twist-two density.

Note once again that the choice (10) corresponds exactly to the Eq. (5), i.e. to the extension of the standard twist-two relation (4b) between F_2 and parton densities at the NLO formulas with the purpose to include the HT contributions.

Note also that for both of the above parton density choices the DGLAP equation will be violated by the HT corrections (see Section VI and discussions therein).

B. Higher twist terms in the renormalon model

As it has been already noted above it is useful to split the parton distributions in two parts

$$f_a(z,Q^2) \;=\; f_a^{+}(z,Q^2) \;+\; f_a^{-}(z,Q^2) \;, \tag{11}$$

where the both $'+'$ and $'-'$ components contain twist-two and HT parts.

The two component representation follows directly form the exact solution of DGLAP equation in the Mellin moment space at the leading twist approximation (see [44]).

The twist-two contribution is presented below in the Section II and the twist-four and twist-six parts can be expressed through the twist-two one as follows (here for simplicity we restrict our consideration by LO approximation): for the (singlet) quark distribution

$$\frac{f_q^{R\tau 4,+}(z,Q^2)}{f_{q,\mathrm{LO}}^{\tau 2,+}(z,Q^2)} \;=\frac{64C_FT_Rf}{15\beta_0^2}\, \mathrm{a}_q^{\tau 4} \left\{ \frac{2}{\rho_{\mathrm{LO}}^2} + \ln\left(\frac{Q^2}{\left|\mathrm{a}_q^{\tau 4}\right|}\right) \frac{\widetilde{I}_0(\sigma_{\mathrm{LO}})}{\rho_{\mathrm{LO}}\, \widetilde{I}_1(\sigma_{\mathrm{LO}})} \right\} \;+\; \mathcal{O}\left(\rho_{\mathrm{LO}}\right) \;, \tag{12a}$$

$$\frac{f_q^{R\tau 4,-}(z,Q^2)}{f_{q,\mathrm{LO}}^{\tau 2,-}(z,Q^2)} \;=\frac{64C_FT_Rf}{15\beta_0^2}\, \mathrm{a}_q^{\tau 4} \left\{ \ln\left(\frac{1}{z_q}\right) \ln\left(\frac{Q^2}{z_q\left|\mathrm{a}_q^{\tau 4}\right|}\right) - p'(\nu_q) \right\} \;+\; \mathcal{O}\left(z\right) \;. \tag{12b}$$

for the gluon distribution

$$\frac{f_G^{R\tau 4,+}(z,Q^2)}{f_{G,\mathrm{LO}}^{\tau 2,+}(z,Q^2)} = \frac{8}{5\beta_0^2}\,\frac{\mathrm{a}_G^{\tau 4}}{a_s(Q^2)}\,\left\{\frac{2}{\rho_{\mathrm{LO}}}\frac{\widetilde{I}_1(\sigma_{\mathrm{LO}})}{\widetilde{I}_0(\sigma_{\mathrm{LO}})} + \ln\left(\frac{Q^2}{|\mathrm{a}_G^{\tau 4}|}\right)\right\} \;+\; \mathcal{O}\left(\rho_{\mathrm{LO}}\right)\;, \tag{12c}$$

$$\frac{f_G^{R\tau 4,-}(z,Q^2)}{f_{G,\mathrm{LO}}^{\tau 2,-}(z,Q^2)} = \frac{8}{5\beta_0^2}\,\frac{\mathrm{a}_G^{\tau 4}}{a_s(Q^2)}\,\ln\left(\frac{Q^2}{z_G^2\,|\mathrm{a}_G^{\tau 4}|}\right) \;+\; \mathcal{O}\left(z\right)\;, \tag{12d}$$

where $\mathrm{a}_a^{\tau 4}$ are the magnitudes which should be extracted from the fits of the experimental data. The variables $z_a = z\exp[p(\nu_a)]$, where $p(\nu_a) = [\Psi(1+\nu_a) - \Psi(\nu_a)]$ and ν_a are the powers of the $x \to 1$ asymptotics of the parton distributions, i. e. $f_a \sim (1-x)^{\nu_a}$ at $x \to 1$. From the quark counting rules we know that $\nu_q \approx 3$ and $\nu_G \approx 4$. Then, we get $p(\nu_q) \approx 11/6$ and $p(\nu_G) \approx 25/12$, and there derivatives $p'(\nu_q) \approx -49/36$ and $p'(\nu_G) \approx -205/144$ (see Appendix B for further details).

The functions $\widetilde{I}_\nu$ in Eqs. (12a, 12c) are related to the modified Bessel function I_ν and to the Bessel function J_ν by:

$$\widetilde{I}_\nu(\sigma) \;=\; \begin{cases} I_\nu(\bar\sigma), & \text{if } \sigma^2 = \bar\sigma^2 \geq 0\;, \\ i^\nu J_\nu(\bar\sigma), & \text{if } \sigma^2 = -\bar\sigma^2 < 0\;. \end{cases} \tag{13}$$

and the σ and ρ values are given in the Section II by Eqs. (20) and (23) at the LO and by Eqs. (26) NLO, respectively.

Note that the upper (down) line in the r.h.s. of Eq. (13) corresponds to the solution of the DGLAP equation for the "direct" ("backward") evolution in the DAS approximation.

The twist-six part can be easy obtained from the corresponding twist-four one as

$$f_a^{R\tau 6}(z,Q^2) \;=\; -\frac{8}{7}\times\left[f_a^{R\tau 4}(z,Q^2)\;\;\text{with}\;\;\mathrm{a}_a^{\tau 4}\to\mathrm{a}_a^{\tau 6}\,,\;\ln\left(\frac{Q^2}{|\mathrm{a}_a^{\tau 4}|}\right)\to\ln\left(\frac{Q^2}{\sqrt{|\mathrm{a}_a^{\tau 6}|}}\right)\right]\;. \tag{14}$$

The article is organized as follows. In Section II we shortly review basic formulae of the solution of DGLAP equation at small x values with the flat initial conditions, given in [44]. We show the possibility to add the backward evolution to the formulae. In Sections III and IV we present the set of formulae for the derivation $\partial F_2/\partial\ln Q^2$ and for the effective slope $\partial\ln F_2/\partial\ln(1/x)$. Section V contains our suggestions about the contributions of twist-four and twist-six operators of the Wilson operator product expansion. In Sections VI–IX we consider the contributions of the HT operators to parton distributions and to derivatives of F_2 in the framework of the infrared renormalon model. Section X contains the fits of experimental data for F_2, predictions for its derivatives and some discussions of the obtained results. In the Appendix A we present Mellin moments of renormalon contributions, calculated in [50], and obtain their contributions to the PD corresponding moments. In the Appendix B we illustrate the method [54, 55] of replacing at small x the convolution of two functions by simple products. The method is used in the present work for the correct incorporation of renormalon-type contributions of higher twists terms into our formulae. The conclusions contains summary of the results and suggestions about other applications of the approach.

II. THE CONTRIBUTION OF TWIST-TWO OPERATORS

As in [44], we will work with the small x asymptotic form of parton distributions in the framework of the DGLAP evolution equations starting at some Q_0^2 with the flat function:

$$f_a(Q_0^2) \;=\; A_a \qquad (\text{hereafter } a=q,G)\;, \tag{15}$$

where A_a are unknown parameters that have to be determined from data.

We shortly compile below the main results found in [44] at the LO NLO approximations.

A. Leading order

The small x asymptotic results for PD, $f_{a,\mathrm{LO}}^{\tau 2}$ $(a=q,G)$ and $F_{2,\mathrm{LO}}^{\tau 2}$ at LO of perturbation theory and at twist-two in the operator product expansion have been found in Ref. [44]:

$$F_{2,\mathrm{LO}}^{\tau 2}(z,Q^2) \;=\; e\; f_{q,\mathrm{LO}}^{\tau 2}(z,Q^2)\;, \tag{16a}$$

$$f_{a,\mathrm{LO}}^{\tau 2}(z,Q^2) \;=\; f_{a,\mathrm{LO}}^{\tau 2,+}(z,Q^2) \;+\; f_{a,\mathrm{LO}}^{\tau 2,-}(z,Q^2)\;. \tag{16b}$$

After Mellin inversion of the explicit moment solution to DGLAP equations, the $'+'$ and $'-'$ PD components are given by:

$$f^{\tau2,+}_{G,\mathrm{LO}}(z,Q^2) = \left(A^{\tau2}_G + \frac{C_F}{C_A}A^{\tau2}_q\right)\, \widetilde{I}_0(\sigma_{\mathrm{LO}})\;\mathrm{e}^{-\bar{d}_+(1)s_{\mathrm{LO}}} \;+\; \mathcal{O}\left(\rho_{\mathrm{LO}}\right)\;, \tag{16c}$$

$$f^{\tau2,+}_{q,\mathrm{LO}}(z,Q^2) = \frac{2T_Rf}{3C_A}\;\left(A^{\tau2}_G + \frac{C_F}{C_A}A^{\tau2}_q\right)\,\rho_{\mathrm{LO}}\;\widetilde{I}_1(\sigma_{\mathrm{LO}})\;\mathrm{e}^{-\bar{d}_+(1)s_{\mathrm{LO}}} \;+\; \mathcal{O}\left(\rho_{\mathrm{LO}}\right)\;, \tag{16d}$$

$$f^{\tau2,-}_{G,\mathrm{LO}}(z,Q^2) = -\frac{C_F}{C_A}\;A^{\tau2}_q\;\mathrm{e}^{-d_-(1)s_{\mathrm{LO}}} \;+\; \mathcal{O}\left(z\right)\;, \tag{16e}$$

$$f^{\tau2,-}_{q,\mathrm{LO}}(z,Q^2) = A^{\tau2}_q\;\mathrm{e}^{-d_-(1)s_{\mathrm{LO}}} \;+\; \mathcal{O}\left(z\right)\;, \tag{16f}$$

where

$$\bar{d}_+(1) = 1 + \frac{8T_Rf}{3\beta_0}\left(1 - \frac{C_F}{C_A}\right)\;,\quad d_-(1) = \frac{8C_FT_Rf}{3C_A\beta_0} \tag{17}$$

are the regular parts of d_+ and d_- anomalous dimensions, respectively, in the limit $n \to 1$. [132]

We define the variable

$$s = \ln\left(\frac{a_s(Q_0^2)}{a_s(Q^2)}\right)\;. \tag{18}$$

At LO, in terms of the QCD scale Λ_{LO}, it has the form:

$$s_{\mathrm{LO}} = \ln\left(\frac{\ln(Q^2/\Lambda^2_{\mathrm{LO}})}{\ln(Q_0^2/\Lambda^2_{\mathrm{LO}})}\right)\;. \tag{19}$$

The argument σ_{LO} in the LO is given by [133]

$$\sigma_{\mathrm{LO}} = 2\sqrt{\hat{d}_{GG}s_{\mathrm{LO}}\ln(z)}\;, \tag{20}$$

where

$$\hat{d}_{GG} = -4C_A/\beta_0 \tag{21}$$

is the singular part when $n \to 1$ of $d_{GG} = \gamma^{(0)}_{GG}(n)/(2\beta_0)$ being $\gamma^{(0)}_{GG}(n)$ the LO coefficient of the gluon-gluon anomalous dimension.

The prescription for the backward evolution given by Eq. (13) is the result, in the more general case, of the following representation of the series which appear in the inverse Mellin transformation of the exact solution for PD moments. (see for example Eq. (6) in Ref. [44]),

$$\sum_{k=0}^{\infty}\frac{t^k}{k!\Gamma(k+\nu+1)} \;=\; t^{-\nu/2}\widetilde{I}_\nu(2\sqrt{t}) \;\equiv\; |t|^{-\nu/2}\begin{cases} I_\nu(2\sqrt{|t|}), & \text{if } t \geq 0\;, \\ J_\nu(2\sqrt{|t|}), & \text{if } t < 0\;. \end{cases} \tag{22}$$

And finally, in Eq. (16d)

$$\rho_{\mathrm{LO}} = \sqrt{\frac{\hat{d}_{GG}s_{\mathrm{LO}}}{\ln(z)}} = \frac{\sigma_{\mathrm{LO}}}{2\ln(1/z)}\;, \tag{23}$$

Let us note, that

$$\rho^{-\nu}\;\widetilde{I}_\nu(\sigma) \;\to\; \frac{1}{\nu!}\;\ln^\nu(1/z) \quad \text{at} \quad Q^2 \;\to\; Q_0^2\;. \tag{24}$$

B. Next-to-leading order

The small x behavior of the twist-2 parton densities $f_a^{\tau 2}$ $(a=q,G)$ and of $F_2^{\tau 2}$ at the NLO approximation has been presented in our previous paper [44]. Here we give the result that can also be used for Q^2 below the initial condition point Q_0^2 (where partons have the flat form in x as Eq. (15))

$$F_2^{\tau 2}(z,Q^2) \;=\; e\left(f_q^{\tau 2}(z,Q^2) \;+\; \frac{4T_Rf}{3}\,a_s(Q^2)\,f_G^{\tau 2}(z,Q^2)\right) \;, \tag{25a}$$

$$f_a^{\tau 2}(z,Q^2) \;=\; f_a^{\tau 2,+}(z,Q^2) \;+\; f_a^{\tau 2,-}(z,Q^2)\;. \tag{25b}$$

The $'+'$ and $'-'$ PD components in the equations above are:

$$f_G^{\tau 2,+}(z,Q^2) \;=A_G^+(Q^2,Q_0^2)\,\widetilde{I}_0(\sigma)\,\exp(-\bar{d}_+(1)s-\bar{D}_+(1)p) \;+\; \mathcal{O}(\rho)\;, \tag{25c}$$

$$\begin{aligned} f_q^{\tau 2,+}(z,Q^2) \;&=A_q^+(Q^2,Q_0^2)\,\left[\left(1-\bar{d}^q_{+-}(1)a_s(Q^2)\right)\,\rho\widetilde{I}_1(\sigma)+\frac{20C_A}{3}\,a_s(Q^2)\,\widetilde{I}_0(\sigma)\right] \\ &\times\exp(-\bar{d}_+(1)s-\bar{D}_+(1)p) \;+\; \mathcal{O}(\rho)\;, \end{aligned} \tag{25d}$$

$$f_a^{\tau 2,-}(z,Q^2) \;=A_a^-(Q^2,Q_0^2)\,\exp(-d_-(1)s-D_-(1)p) \;+\; \mathcal{O}(z)\;, \tag{25e}$$

where $D_\pm(n)=d_{\pm\pm}(n)-(\beta_1/\beta_0)d_\pm(n)$; $p=a_s(Q_0^2)-a_s(Q^2)$ and

$$\sigma=2\sqrt{(\hat{d}_+s+\hat{D}_+p)\ln(z)}\;\;,\qquad \rho=\sqrt{\frac{(\hat{d}_+s+\hat{D}_+p)}{\ln(z)}}=\frac{\sigma}{2\ln(1/z)}\;. \tag{26}$$

$$A_G^+(Q^2,Q_0^2)=\left[1-\bar{d}^G_{+-}(1)a_s(Q^2)\right]A_G^{\tau 2}+\frac{C_F}{C_A}\left[1-d^G_{-+}(1)a_s(Q_0^2)-\bar{d}^G_{+-}(1)a_s(Q^2)\right]A_q^{\tau 2}\;, \tag{27a}$$

$$A_G^-(Q^2,Q_0^2)=A_G^{\tau 2}-A_G^+(Q^2,Q_0^2)\;, \tag{27b}$$

$$A_q^+(Q^2,Q_0^2)=\frac{2T_Rf}{3C_A}\left(A_G^{\tau 2}+\frac{C_F}{C_A}A_q^{\tau 2}\right)\;, \tag{27c}$$

$$A_q^-(Q^2,Q_0^2)=A_q^{\tau 2}-\frac{20C_A}{3}\,a_s(Q_0^2)\,A_q^+(Q^2,Q_0^2)\;. \tag{27d}$$

The different singular and regular parts of anomalous dimensions appearing in Eqs. (25)–(26) have the form [134]:

$$\hat{d}_{++}=\frac{8T_Rf}{9\beta_0}\,(23C_A-26C_F)\;\;,\;\;\hat{d}^q_{+-}=-\frac{20C_A}{3}\;\;,\;\;\hat{d}^G_{+-}=0\;, \tag{28a}$$

$$\begin{aligned} \bar{d}_{++}(1)=\frac{8}{3\beta_0}&\left[\frac{C_A^2}{3}\left(36\zeta(3)+33\zeta(2)-\frac{1643}{12}\right)\right. \\ &\left.-\left(4C_F\zeta(2)+\frac{86}{9}C_A-\frac{547}{18}C_F+3\frac{C_F^2}{C_A}\right)T_Rf-\frac{26C_F}{9C_A}\left(1-2\frac{C_F}{C_A}\right)T_R^2f^2\right]\;, \end{aligned} \tag{28b}$$

$$\bar{d}^q_{+-}(1)=C_A\left(9-3\frac{C_F}{C_A}-4\zeta(2)\right)-\frac{26}{9}\left(1-2\frac{C_F}{C_A}\right)T_Rf\;,\;\;\bar{d}^G_{+-}(1)=\frac{40C_FT_Rf}{9C_A}\;, \tag{28c}$$

$$\begin{aligned} d_{--}(1)=&\frac{4C_AC_F}{\beta_0}\left(1-2\frac{C_F}{C_A}\right)\left(2\zeta(3)-3\zeta(2)+\frac{13}{4}+\frac{52T_R^2f^2}{27C_A^2}\right) \\ &+\frac{8C_F}{3\beta_0}\left(4\zeta(2)-\frac{47}{18}+3\frac{C_F}{C_A}\right)T_Rf\;, \end{aligned} \tag{28d}$$

$$d^q_{-+}(1)=0\;,\;\;d^G_{-+}(1)=-\left(C_A+\frac{2}{3}\left(1-2\frac{C_F}{C_A}\right)T_Rf\right)\;. \tag{28e}$$

The corresponding numerical values are collected in Table I (see Ref. [44] for details).

We would like to note that the exact value of the variable σ and the small x asymptotics of the modified Bessel function

$$I_\nu(\sigma)\;\sim\;\exp\left(\sigma\right)\quad\text{at}\quad\sigma\to\infty$$

TABLE I: The values of the parameters used in the calculation of the parton distributions as a function of the number of flavors.

f	$\hat{d}_+$	$\hat{D}_+$	$\bar{d}_+(1)$	$\bar{D}_+(1)$	$d_-(1)$	$D_-(1)$	$\bar{d}^q_{+-}(1)$	$\bar{d}^G_{+-}(1)$	$d^G_{-+}(1)$
3	−4/3	1180/81	101/81	−43.370269	16/81	1.974431	2.779310	80/27	−29/9
4	−36/25	91096/5625	61/45	−45.485532	64/225	3.108220	2.618816	320/81	−89/27
5	−36/23	84964/4761	307/207	−47.729779	80/207	4.674958	2.458322	400/81	−91/27
6	−12/7	8576/441	103/63	−50.057345	32/63	6.864360	2.297828	160/27	−31/9

are given in Introduction (see Eq. (1)) with $|\hat{d}_+| = a_+$ and $\hat{D}_+ = b_+ + a_+(\beta_1/\beta_0)$. So, the most important part from the NLO corrections (i.e. the singlet part at $x \to 0$) is taken in a proper way: it comes directly into the argument of the Bessel functions and does not spoil the applicability of perturbation theory at low x values.

We stress that the LO and NLO results given above coincide with the ones in Ref. [44] for positive values of s and s_{LO} (i.e. for the case $Q^2 \geq Q_0^2$).

Let us remind that these analytical expressions which have been obtained from the exact solution to the moment space DGLAP evolution equations in the asymptotic limit $n \to 1$ have been already used in Ref. [44] to reproduce the small x behavior of parton distributions and lastly of DIS structure functions themselves. The consideration of negative values for s and s_{LO} leads us to apply the backward evolution in the present analysis and, thus, to have the possibility to choose any normalization point Q_0^2 and not only the low end of the Q^2-evolution as it was done in Ref. [44].

III. THE CONTRIBUTION OF TWIST-TWO OPERATORS TO THE DERIVATIVE $\partial F_2/\partial \ln Q^2$

In QCD the scaling violation of $F_2(z,Q^2)$ are caused by gluon bremsstahlung from quarks and quark pair creation from gluons. In the low x domain the latter process dominates the scaling violations. F_2 is then largely determined by the sea quarks, whereas the $\partial F_2/\partial \ln Q^2$ is dominated by the convolution of the splitting function P_{qG} and the gluon density. At the leading twist approximation the derivative $\partial F_2/\partial \ln Q^2$ relates strongly to the gluon distribution $f_G^{\tau 2}(z,Q^2)$. Moreover, the derivative is measured with a good accuracy. Then, the $\partial F_2/\partial \ln Q^2$ experimental data can be successfully used to determine the characteristic properties of gluon distribution.

The $\partial F_2/\partial \ln Q^2$ data becomes even more important, when we add higher twist corrections into consideration. In the case of the twist-four terms (of sum of the twist-four and twist-six terms) in the renormalon model there are the two (four) additional parameters (see below Section IV) which may lead to problems to fit all them together only with help of F_2 experimental data.

A. Leading order

Note that at the LO approximation there are the following properties

$$\frac{\partial}{\partial \ln Q^2}\left[\frac{1}{\rho_{\rm LO}^k}\widetilde{I}_k(\sigma_{\rm LO})\right] = 4C_A\, a_s(Q^2)\,\frac{1}{\rho_{\rm LO}^{k+1}}\widetilde{I}_{k+1}(\sigma_{\rm LO}) \,, \tag{29a}$$

$$\frac{\partial}{\partial \ln Q^2}\left[\rho_{\rm LO}^k\widetilde{I}_k(\sigma_{\rm LO})\right] = 4C_A\, a_s(Q^2)\,\rho_{\rm LO}^{k-1}\widetilde{I}_{|k-1|}(\sigma_{\rm LO}) \quad (k=0,1,2,\ldots) \,, \tag{29b}$$

which lead to the following results

$$\frac{\partial f^{\tau 2,+}_{G,{\rm LO}}(z,Q^2)}{\partial \ln Q^2} = a_s(Q^2)\left[\frac{4C_A}{\rho_{\rm LO}}\,\frac{\widetilde{I}_1(\sigma_{\rm LO})}{\widetilde{I}_0(\sigma_{\rm LO})} - \beta_0\,\bar{d}_+(1)\right] f^{\tau 2,+}_{G,{\rm LO}}(z,Q^2) \;+\; \mathcal{O}(\rho_{\rm LO}) \,, \tag{30a}$$

$$\frac{\partial f^{\tau 2,+}_{q,{\rm LO}}(z,Q^2)}{\partial \ln Q^2} = a_s(Q^2)\left[\frac{8T_Rf}{3}\, f^{\tau 2,+}_{G,{\rm LO}}(z,Q^2) - \beta_0\,\bar{d}_+(1) f^{\tau 2,+}_{q,{\rm LO}}(z,Q^2)\right] \;+\; \mathcal{O}(\rho_{\rm LO}) \,, \tag{30b}$$

$$\frac{\partial f^{\tau 2,-}_{G,{\rm LO}}(z,Q^2)}{\partial \ln Q^2} = -a_s(Q^2)\,\frac{8C_FT_Rf}{3C_A}\, f^{\tau 2,-}_{G,{\rm LO}}(z,Q^2) \;+\; \mathcal{O}(z) \,, \tag{30c}$$

$$\frac{f^{\tau 2,-}_{q,{\rm LO}}(z,Q^2)}{\partial \ln Q^2} = a_s(Q^2)\,\frac{8T_Rf}{3}\, f^{\tau 2,-}_{G,{\rm LO}}(z,Q^2) \;+\; \mathcal{O}(z) \,. \tag{30d}$$

Thus, we have

$$\frac{\partial F^{\tau 2}_{2,\rm LO}(z,Q^2)}{\partial \ln Q^2} \;=\; e\, \frac{\partial f^{\tau 2}_{q,\rm LO}(z,Q^2)}{\partial \ln Q^2} \;=\; e\, a_s(Q^2) \left[\frac{8T_R f}{3}\, f^{\tau 2}_{G,\rm LO}(z,Q^2) - \beta_0\, \bar{d}_+(1) f^{\tau 2,+}_{q,\rm LO}(z,Q^2)\right] \tag{31}$$

The LO Q^2 evolution of the derivative $\partial F_2^{\tau 2}/\partial \ln Q^2$ is defined mostly by the corresponding evolution of the gluon distribution $f^{\tau 2}_{G,\rm LO}(z,Q^2)$, i.e. by the Eqs. (16b, 16c) and (16e).

B. Next-to-leading order

At the NLO approximation of perturbation theory the Eqs. (29) are replaced by

$$\begin{aligned}
\frac{\partial}{\partial \ln Q^2}\left[\frac{1}{\rho^k}\widetilde{I}_k(\sigma)\right] \;&=\; a_s(Q^2)\left[4C_A - a_s(Q^2)\beta_0 \hat{d}_{++}\right] \frac{1}{\rho^{k+1}}\widetilde{I}_{k+1}(\sigma)\;, && (32a)\\
\frac{\partial}{\partial \ln Q^2}\left[\rho^k\widetilde{I}_k(\sigma)\right] \;&=\; a_s(Q^2)\left[4C_A - a_s(Q^2)\beta_0 \hat{d}_{++}\right] \rho^{k-1}\widetilde{I}_{|k-1|}(\sigma) \quad (k=0,1,2,\dots)\;, && (32b)
\end{aligned}$$

which leads to the following results

$$\begin{aligned}
\frac{\partial f_q^{\tau 2,+}(z,Q^2)}{\partial \ln Q^2} &= a_s(Q^2)\,\frac{2T_R f}{3C_A}\,\left(A_G^{\tau 2} + \frac{C_F}{C_A}A_q^{\tau 2}\right)\,\Bigg[4C_A\widetilde{I}_0(\sigma) \;-\; \beta_0\,\bar{d}_+(1)\rho\,\widetilde{I}_1(\sigma)\\
&+ a_s(Q^2)\left\{\frac{80}{3}\,\frac{C_A^2}{\rho}\,\widetilde{I}_1(\sigma) \;-\; \left(\beta_0\left[\hat{d}_{++} \;+\; \frac{20}{3}\,C_A\Big(1+\bar{d}_+(1)\Big)\right] \;-\; 4C_A\bar{d}^q_{+-}(1)\right)\widetilde{I}_0(\sigma)\right.\\
&+\;\left.\beta_0\left(\bar{d}^q_{+-}(1)\Big(1+\bar{d}_+(1)\Big) - \bar{d}_{++}(1)\right)\rho\,\widetilde{I}_1(\sigma)\right\}\Bigg]\; \exp(-\bar{d}_+(1)s - \bar{D}_+(1)p) \;+\; \mathcal{O}(\rho)\;, && (33)\\
\frac{f_q^{\tau 2,-}(z,Q^2)}{\partial \ln Q^2} &= -\beta_0\, a_s(Q^2)\,\Bigg[A_q^{\tau 2}\Big(d_-(1) \;+\; a_s(Q^2)\,d_{--}(1)\Big)\\
&-\;\frac{40T_R f}{9}\,a_s(Q_0^2)\,d_-(1)\,\left(A_G^{\tau 2} + \frac{C_F}{C_A}A_q^{\tau 2}\right)\Bigg]\;\exp(-d_-(1)s - D_-(1)p) \;+\; \mathcal{O}(z)\;. && (34)
\end{aligned}$$

Taking together equations (25), (30a), (30c), (33) and (34), after some algebra we have got the final result

$$\begin{aligned}
\frac{\partial F_2^{\tau 2}(z,Q^2)}{\partial \ln Q^2} \;&=\; e\, a_s(Q^2)\left[\frac{8T_R f}{3}\left(f_G^{\tau 2}(z,Q^2) \;+\; \Phi(z,Q^2)\right) \;-\; \beta_0\,\bar{d}_+(1) f_q^{\tau 2,+}(z,Q^2)\right.\\
&\left.-\; a_s(Q^2)\,\beta_0\,d_{--}(1) f_q^{\tau 2,-}(z,Q^2)\right]\;, && (35)
\end{aligned}$$

where

$$\begin{aligned}
\Phi(z,Q^2) \;&=\; \Phi^+(z,Q^2) \;+\; \Phi^-(z,Q^2)\;, && (36a)\\
\Phi^+(z,Q^2) \;&=\; \phi^+(z,Q^2)\,\exp(-\bar{d}_+(1)s - \bar{D}_+(1)p) \;+\; \mathcal{O}(\rho)\;, && (36b)\\
\Phi^-(z,Q^2) \;&=\; \phi^-(z,Q^2)\,\exp(-d_-(1)s - D_-(1)p) \;+\; \mathcal{O}(z)\;. && (36c)
\end{aligned}$$

The $'+'$ and $'-'$ components in the equations above are:

$$\begin{aligned}
\phi^+(z,Q^2) \;&=\; a_s(Q^2)\,\left(A_G^{\tau 2} + \frac{C_F}{C_A}A_q^{\tau 2}\right)\,\left\{\frac{26}{3}\,\frac{C_A}{\rho}\,\widetilde{I}_1(\sigma) \;-\; \left(\frac{\beta_0}{4C_A}\left[\hat{d}_{++} \;+\; \frac{2}{3}\,C_A\Big(13+3\bar{d}_+(1)\Big)\right] \;+\; \bar{d}^q_{+-}(1)\right.\right.\\
&\left.\left.-\;\bar{d}^G_{+-}(1)\right)\widetilde{I}_0(\sigma) \;+\; \frac{\beta_0}{4C_A}\left(\bar{d}^q_{+-}(1) - \bar{d}_{++}(1)\right)\rho\,\widetilde{I}_1(\sigma)\right\} \;+\; a_s(Q_0^2)\,A_q^{\tau 2}\,d^G_{-+}(1)\,\widetilde{I}_0(\sigma)\;, && (36d)\\
\phi^-(z,Q^2) \;&=\; \Big(a_s(Q_0^2) \;-\; a_s(Q^2)\Big)\left\{\bar{d}^G_{+-}(1)\,\left(A_G^{\tau 2} + \frac{C_F}{C_A}A_q^{\tau 2}\right) \;-\; d^G_{-+}(1)\,\frac{C_F}{C_A}A_q^{\tau 2}\right\} + \;\frac{17C_F}{6}\,a_s(Q^2)\,A_q^{\tau 2}\;. && (36e)
\end{aligned}$$

The values of the coefficients are given in Eqs. (28).

Thus, the NLO Q^2 evolution of the derivative $\partial F_2^{\tau 2}/\partial \ln Q^2$ is defined mostly by the corresponding evolution of the gluon distribution $f_G^{\tau 2}(z,Q^2)$, i.e. by the Eqs. (25b, 25c) and (25e).

IV. THE CONTRIBUTION OF TWIST-TWO OPERATORS TO THE SLOPES OF F_2 AND OF PARTON DISTRIBUTIONS

The behavior of F_2 and parton distributions can mimic a power law shape over a limited region of z, Q^2:

$$f_a(z,Q^2) \sim z^{-\lambda^{\rm eff}_a(z,Q^2)} \quad \text{and} \quad F_2(z,Q^2) \sim z^{-\lambda^{\rm eff}_{F2}(z,Q^2)} \,. \tag{37}$$

The slopes are effective ones because the parton distributions and F_2 have mostly the Bessel-like form.

Note that there are the following properties

$$\frac{\partial}{\partial \ln(1/z)}\left[\frac{1}{\rho^k}\widetilde{I}_k(\sigma)\right] \;=\; \frac{1}{\rho^{k-1}}\widetilde{I}_{k-1}(\sigma)\;, \tag{38a}$$

$$\frac{\partial}{\partial \ln(1/z)}\left[\rho^k\widetilde{I}_k(\sigma)\right] \;=\; \rho^{k+1}\widetilde{I}_{k+1}(\sigma) \quad (k=0,1,2,\ldots)\;, \tag{38b}$$

which we will use below.

A. Leading order

The effective slopes have the form at the LO approximation

$$\lambda^{\rm eff,\tau2}_{G,{\rm LO}}(z,Q^2) \;=\; \frac{f^{\tau2,+}_{G,{\rm LO}}(z,Q^2)}{f^{\tau2}_{G,{\rm LO}}(z,Q^2)}\,\rho_{\rm LO}\,\frac{\widetilde{I}_1(\sigma_{\rm LO})}{\widetilde{I}_0(\sigma_{\rm LO})}\;, \tag{39a}$$

$$\lambda^{\rm eff,\tau2}_{F2,{\rm LO}}(z,Q^2) \;=\; \lambda^{\rm eff,\tau2}_{q,{\rm LO}}(z,Q^2) \;=\; \frac{f^{\tau2,+}_{q,{\rm LO}}(z,Q^2)}{f^{\tau2}_{q,{\rm LO}}(z,Q^2)}\,\rho_{\rm LO}\,\frac{\widetilde{I}_2(\sigma_{\rm LO})}{\widetilde{I}_1(\sigma_{\rm LO})}\;. \tag{39b}$$

The effective slopes $\lambda^{\rm eff}_a$ and $\lambda^{\rm eff}_{F2}$ depend on the magnitudes $A^{\tau2}_a$ of the initial PD and also on the chosen input values of Q^2_0 and Λ. At quite large values of $Q^2 >> Q^2_0$, where the '$-$' component is not relevant, the dependence on the magnitudes of the initial PD disappear, having in this case for the asymptotic values:

$$\lambda^{\rm eff,\tau2}_{G,{\rm LO,as}}(z,Q^2) \;=\rho_{\rm LO}\,\frac{\widetilde{I}_1(\sigma_{\rm LO})}{\widetilde{I}_0(\sigma_{\rm LO})} \approx \rho_{\rm LO} - \frac{1}{4\ln(1/z)}\;, \tag{40a}$$

$$\lambda^{\rm eff,\tau2}_{F2,{\rm LO,as}}(z,Q^2) \;=\lambda^{\rm eff,\tau2}_{q,{\rm LO,as}}(z,Q^2) \;=\; \rho_{\rm LO}\,\frac{\widetilde{I}_2(\sigma_{\rm LO})}{\widetilde{I}_1(\sigma_{\rm LO})} \approx \rho_{\rm LO} - \frac{3}{4\ln(1/z)}\;, \tag{40b}$$

where symbol $\approx$ marks approximations obtained by expansions of modified Bessel functions $I_n(\sigma)$. These approximations should be correct only at very large σ values (i.e. at very large Q^2 and/or very small x). It is the case (see Figs. 2 and 6).

We would like to note that the slope $\lambda^{\rm eff,\tau2}_{F2,{\rm LO,as}}(z,Q^2) = \lambda^{\rm eff,\tau2}_{q,{\rm LO,as}}(z,Q^2)$ coincides at very large σ with one obtained in [57] (see also [15]) in the case of flat input. Note that the slope $\lambda^{\rm eff,\tau2}_{G,{\rm LO,as}}(z,Q^2)$ is large then the slope $\lambda^{\rm eff,\tau2}_{F2,{\rm LO,as}}(z,Q^2) = \lambda^{\rm eff,\tau2}_{q,{\rm LO,as}}(z,Q^2)$:

$$\lambda^{\rm eff,\tau2}_{G,{\rm LO,as}}(z,Q^2) - \lambda^{\rm eff,\tau2}_{F2,{\rm LO,as}}(z,Q^2) \;=\; \rho_{\rm LO}\left(\frac{\widetilde{I}_1(\sigma_{\rm LO})}{\widetilde{I}_0(\sigma_{\rm LO})} - \frac{\widetilde{I}_2(\sigma_{\rm LO})}{\widetilde{I}_1(\sigma_{\rm LO})}\right) \approx \frac{1}{2\ln(1/z)}\;, \tag{41}$$

that coincides with results of fits in Refs. [30, 34].

B. Next-to-leading order

At the NLO approximation of perturbation theory we have the following properties of the effective slopes: the quark and gluon ones $\lambda^{\rm eff,\tau2}_a(z,Q^2) = \partial \ln f^{\tau2}_a(z,Q^2)/\partial \ln(1/z)$ are reduced by the NLO terms that leads to the decreasing of the gluon distribution at small x. For the quark case it is not the case, because the normalization factor $A^{\tau2,+}_q$ of the '+' component produces an additional contribution undamped as $\sim (\ln z)^{-1}$.

Indeed, the effective slopes have the form,

$$\lambda_G^{\mathrm{eff},\tau2}(z,Q^2) = \frac{f_G^{\tau2,+}(z,Q^2)}{f_G^{\tau2}(z,Q^2)}\,\rho\,\frac{\widetilde{I}_1(\sigma)}{\widetilde{I}_0(\sigma)}\ , \tag{42a}$$

$$\lambda_q^{\mathrm{eff},\tau2}(z,Q^2) = \frac{f_q^{\tau2,+}(z,Q^2)}{f_q^{\tau2}(z,Q^2)}\,\rho\,\frac{\widetilde{I}_2(\sigma)\left(1-\bar{d}_{+-}^q(1)a_s(Q^2)\right)+(20C_A/3)a_s(Q^2)\widetilde{I}_1(\sigma)/\rho}{\widetilde{I}_1(\sigma)\left(1-\bar{d}_{+-}^q(1)a_s(Q^2)\right)+(20C_A/3)a_s(Q^2)\widetilde{I}_0(\sigma)/\rho}\ , \tag{42b}$$

$$\lambda_{F2}^{\mathrm{eff},\tau2}(z,Q^2) = \frac{\lambda_q^{\mathrm{eff}}(z,Q^2)\,f_q^{\tau2}(z,Q^2)+(4T_Rf/3)\,a_s(Q^2)\,\lambda_G^{\mathrm{eff}}(z,Q^2)\,f_G^{\tau2}(z,Q^2)}{f_q^{\tau2}(z,Q^2)+(4T_Rf/3)\,a_s(Q^2)\,f_G^{\tau2}(z,Q^2)}\ . \tag{42c}$$

The gluon effective slope $\lambda_G^{\mathrm{eff},\tau2}(z,Q^2)$ is larger than the quark slope $\lambda_q^{\mathrm{eff},\tau2}(z,Q^2)$, which is in excellent agreement with a recent MRS and GRV analysis [30, 34].

For the asymptotic values we have got

$$\lambda_{G,\mathrm{as}}^{\mathrm{eff},\tau2}(z,Q^2) = \rho\,\frac{\widetilde{I}_1(\sigma)}{\widetilde{I}_0(\sigma)} \approx \rho-\frac{1}{4\ln(1/z)}\ , \tag{43a}$$

$$\begin{aligned}\lambda_{q,\mathrm{as}}^{\mathrm{eff},\tau2}(z,Q^2) &= \rho\,\frac{\widetilde{I}_2(\sigma)\left(1-\bar{d}_{+-}^q(1)a_s(Q^2)\right)+(20C_A/3)a_s(Q^2)\widetilde{I}_1(\sigma)/\rho}{\widetilde{I}_1(\sigma)\left(1-\bar{d}_{+-}^q(1)a_s(Q^2)\right)+(20C_A/3)a_s(Q^2)\widetilde{I}_0(\sigma)/\rho}\\ &= \rho\,\frac{\widetilde{I}_2(\sigma)}{\widetilde{I}_1(\sigma)}+\frac{20C_A}{3}\alpha(Q^2)\left(1-\frac{\widetilde{I}_0(\sigma)\widetilde{I}_2(\sigma)}{\widetilde{I}_1^2(\sigma)}\right) \approx \rho-\frac{3}{4\ln(1/z)}+\frac{10C_A}{3}\frac{a_s(Q^2)}{\rho\ln(1/z)}\ ,\end{aligned} \tag{43b}$$

$$\begin{aligned}\lambda_{F2,\mathrm{as}}^{\mathrm{eff},\tau2}(z,Q^2) &= \rho\,\frac{\widetilde{I}_2(\sigma)}{\widetilde{I}_1(\sigma)}+\frac{26C_A}{3}\alpha(Q^2)\left(1-\frac{\widetilde{I}_0(\sigma)\widetilde{I}_2(\sigma)}{\widetilde{I}_1^2(\sigma)}\right) = \lambda_{q,\mathrm{as}}^{\mathrm{eff},\tau2}(z,Q^2)+2C_A\,a_s(Q^2)\left(1-\frac{\widetilde{I}_0(\sigma)\widetilde{I}_2(\sigma)}{\widetilde{I}_1^2(\sigma)}\right)\\ &\approx \rho-\frac{3}{4\ln(1/z)}+\frac{13C_A}{3}\frac{a_s(Q^2)}{\rho\ln(1/z)} = \lambda_{q,\mathrm{as}}^{\mathrm{eff},\tau2}(z,Q^2)+\frac{C_Aa_s(Q^2)}{\rho\ln(1/z)}.\end{aligned} \tag{43c}$$

We would like to note that at the NLO approximation the slope $\lambda_{F2,\mathrm{as}}^{\mathrm{eff},\tau2}(z,Q^2)$ lies between quark and gluon ones but closely to quark slope $\lambda_{q,\mathrm{as}}^{\mathrm{eff},\tau2}(z,Q^2)$, that is in agreement with Refs. [30, 34].

Indeed,

$$\begin{aligned}\lambda_{G,\mathrm{as}}^{\mathrm{eff},\tau2}(z,Q^2) - \lambda_{F2,\mathrm{as}}^{\mathrm{eff},\tau2}(z,Q^2) &= \left(\rho\,\frac{\widetilde{I}_1(\sigma)}{\widetilde{I}_0(\sigma)}+\frac{26C_A}{3}a_s(Q^2)\right)\left(1-\frac{\widetilde{I}_0(\sigma)\widetilde{I}_2(\sigma)}{\widetilde{I}_1^2(\sigma)}\right)\\ &\approx \left(\rho-\frac{1}{4\ln(1/z)}+\frac{26C_A}{3}a_s(Q^2)\right)\frac{1}{2\rho\ln(1/z)}\ ,\end{aligned} \tag{44a}$$

$$\lambda_{F2,\mathrm{as}}^{\mathrm{eff},\tau2}(z,Q^2) - \lambda_{q,\mathrm{as}}^{\mathrm{eff},\tau2}(z,Q^2) = 2C_A\,a_s(Q^2)\left(1-\frac{\widetilde{I}_0(\sigma)\widetilde{I}_2(\sigma)}{\widetilde{I}_1^2(\sigma)}\right) \approx \frac{C_Aa_s(Q^2)}{\rho\ln(1/z)}\ . \tag{44b}$$

Both slopes $\lambda_a^{\mathrm{eff},\tau2}(z,Q^2)$ decrease with decreasing z. A z dependence of the slope should not appear for a PD within a Regge type asymptotic ($x^{-\lambda}$) and precise measurement of the slope $\lambda_a^{\mathrm{eff},\tau2}(z,Q^2)$ may lead to the possibility to verify the type of small x asymptotics of parton distributions. The present data, however, are not enough to distinguish this slow x-dependence of $\lambda_a^{\mathrm{eff},\tau2}(z,Q^2)$ (see Fig. 2).

In the following Sections we study the higher-twist contributions to $F_2(x,Q^2)$, its derivatives and parton distributions.

V. THE HIGHER TWIST CONTRIBUTIONS FOR F_2

In the Section we consider two different representations for twist-four effects. The first one comes from Regge-like analysis [46–49]. Thus, it should have right asymptotics at $x\to 0$ limit, but, unfortunately, the knowledge of its form is very restricted.

The second one is based on the IR-renormalon model. The predictions can not reproduce the exact form of $x\to 0$ asymptotics, calculated in Ref. [46–49] but gives rather good agreement with modern experimental data from HERA (see Section X). We think this agreement is similar to one (see Ref. [15]) at larger Q^2 values between DGLAP approach (even for its analytical simplification: the generalized DAS approach [44]) and experiment.

We would like to note here that in the analysis of experimental data performed below we consider both LO and NLO approximations in the twist-two case and for HT corrections in the renormalon case. In the BFKL-motivated approach, for simplicity [135] we restrict the calculation of the HT contribution to the consideration of LO Q^2 evolution alone.

A. BFKL-motivated estimations for twist-four operators

Twist-four operators are known [58] to have their own evolution equations but the diagonalization of the operator anomalous dimensions matrix is a very complicate problem. For our purpose, however, as the relevant limit is $n \to 1$, one can apply the results of Refs. [46–49], which have very simple form and are given in the classical DAS asymptotics considered in Section 2 of Ref. [44]

Here we show that the contribution from twist-four operators can be represented in the same form as the twist-two operators by using the twist-four anomalous dimensions instead of the twist-two ones.

For the singular part of twist-four anomalous dimensions we consider from Ref. [46] the result:

$$\gamma^{\tau 4}_{GG}(n-1) \;=\; 2\,\gamma^{(0)}_{GG}((n-1)/2)\;(1+\varepsilon)\;, \tag{45}$$

where ε is very small: $\varepsilon = 1/1224$.

Eq. (45) allows us to find the relation between the singular part of twist-four operators anomalous dimensions, $\gamma^{\tau 4}_{ab}(n)$ and $\gamma^{\tau 4}_{\pm}(n)$, with the twist-two ones, $\gamma^{(0)}_{ab}(n)$ and $\gamma^{(0)}_{\pm}(n)$. It leads to the following relations:

$$\hat{d}^{\tau 4}_{+} \;=\; \hat{d}^{\tau 4}_{GG} \;=\; a^2 \hat{d}_{+} \;=\; a^2 \hat{d}_{GG}\;, \qquad \hat{d}^{\tau 4}_{-} \;=\; a^2 \hat{d}_{-} \;=\; 0\;, \tag{46}$$

where $a^2 = 4(1+\varepsilon)$ and $\hat{d}_{+} = \hat{d}_{GG}$ is given by Eq. (21).

The prediction for the regular parts $\overline{d}^{\tau 4}_{+}(n)$ and $d^{\tau 4}_{-}(n)$ can not be obtained from Eq. (45), but it should be essentially less important in the kinematical range studied below. Then, in the analysis presented below, we proceed by fixing this non-singular part by means of a relation similar to Eq. (46):

$$\bar{d}^{\tau 4}_{+}(1) \;=\; b\,\bar{d}_{+}(1), \qquad d^{\tau 4}_{-}(1) \;=\; b\,d_{-}(1), \tag{47}$$

and further we examine different "natural" choices of b: $b = 0, 1$ and $a^2/2$.

Note that the non-singular (when $n \to 1$) parts $\bar{d}^{\tau 4}_{+}(1)$, $d^{\tau 4}_{-}(1)$ and $\bar{d}_{+}(1)$, $d_{-}(1)$ determine the behavior of parton distributions and DIS structure functions at non-small x values. Usually fits to experimental data at intermediate and large values of x are performed with the help of the following forms for the structure function F_2:

$$F_2(x,Q^2) \;=\; F^{\tau 2}_2(x,Q^2) \;+\; \frac{1}{Q^2}\,F^{\tau 4}_2(x) \quad \text{or} \tag{48}$$

$$F_2(x,Q^2) \;=\; F^{\tau 2}_2(x,Q^2)\left(1 \;+\; \frac{1}{Q^2}\,f^{\tau 4}_2(x)\right) \tag{49}$$

with Q^2-independent functions $F^{\tau 4}_2(x)$ or $f^{\tau 4}_2(x)$.

In fact Eq. (48) is closed to our choice $b = 0$, i.e. the twist-4 contribution does not evolve logarithmically with Q^2. Also Eq. (49) is analogous to the choice $b = 1$, i.e. twist-two and twist-four operators have the same logarithmic Q^2-dependence at large and intermediate x values. Lastly, the choice $b = a^2/2$ corresponds to the hypothese about applicability of Eq. (45), obtained in the classical DAS limit, to a more wide generalized DAS approximation considered here.

By analogy with Section II we represent the twist-four contribution split in the $'+'$ and $'-'$ parts:

$$F^{\tau 4}_2(z,Q^2) \;=\; e\,f^{\tau 4}_q(z,Q^2)\;, \tag{50a}$$

$$f^{\tau 4}_a(z,Q^2) \;=\; f^{\tau 4,+}_a(z,Q^2) \;+\; f^{\tau 4,-}_a(z,Q^2)\;. \tag{50b}$$

The $'+'$ and $'-'$ PD components are:

$$f^{\tau 4,+}_G(z,Q^2) = \left(A^{\tau 4}_G + \frac{C_F}{C_A}A^{\tau 4}_q\right)\widetilde{I}_0(a\,\sigma_{\rm LO})\,{\rm e}^{-b\bar{d}_{+}(1)s_{\rm LO}} \;+\; \mathcal{O}\left(\rho_{\rm LO}\right)\;, \tag{50c}$$

$$f^{\tau 4,+}_q(z,Q^2) = \frac{2T_R f}{3C_A}\left(A^{\tau 4}_G + \frac{C_F}{C_A}A^{\tau 4}_q\right)\frac{b}{a}\rho_{\rm LO}\,\widetilde{I}_1(a\,\sigma_{\rm LO})\,{\rm e}^{-b\bar{d}_{+}(1)s_{\rm LO}} \;+\; \mathcal{O}\left(\rho_{\rm LO}\right)\;, \tag{50d}$$

$$f^{\tau 4,-}_G(z,Q^2) = -\frac{C_F}{C_A}\,A^{\tau 4}_q\,{\rm e}^{-bd_{-}(1)s_{\rm LO}} \;+\; \mathcal{O}\left(z\right)\;, \tag{50e}$$

$$f^{\tau 4,-}_q(z,Q^2) = A^{\tau 4}_q\,{\rm e}^{-bd_{-}(1)s_{\rm LO}} \;+\; \mathcal{O}\left(z\right)\;, \tag{50f}$$

because the corresponding twist-four projectors (see [59]) have the following form [136]:

$$
\begin{aligned}
\varepsilon^{\tau 4,+}_{qq} &= \varepsilon^{\tau 4,-}_{GG} = \varepsilon^{+}_{qq} \frac{b}{a^2} \quad , \quad \varepsilon^{\tau 4,-}_{aa} = 1 - \varepsilon^{\tau 4,+}_{aa} \; , \\
\varepsilon^{\tau 4,\pm}_{qG} &= \varepsilon^{\pm}_{qG} \frac{b}{a^2} \quad , \quad \varepsilon^{\tau 4,\pm}_{Gq} = \varepsilon^{\pm}_{Gq} \; .
\end{aligned} \tag{51}
$$

In Eqs. (50c–50f) the twist-four parameters $A^{\tau 4}_a$ $(a = q, G)$ have to be determined from fits to experimental data. The full contribution (i.e. the sum of twist-two and twist-four parts) is given by:

$$
f_a(z, Q^2) = f^{\tau 2}_a(z, Q^2) + \frac{1}{Q^2} f^{\tau 4}_a(z, Q^2) \quad \text{and} \tag{52}
$$

$$
F_2(z, Q^2) = F^{\tau 2}_2(z, Q^2) + \frac{1}{Q^2} F^{\tau 4}_2(z, Q^2) \; , \tag{53}
$$

where the leading twist contributions $f^{\tau 2}_a(z, Q^2)$ and $F^{\tau 2}_2(z, Q^2)$ are given at LO by Eqs. (16a,16b) and at NLO by Eqs. (25a,25b).

B. Renormalon model predictions for twist-four operators

The full small x asymptotic results for parton densities and F_2 structure function in the framework of the infrared renormalon model, i.e. F^R_2, at LO of perturbation theory in the twist-four part:

$$
F^R_2(z, Q^2) = F^{\tau 2}_2(z, Q^2) + \frac{1}{Q^2} F^{R\tau 4}_2(z, Q^2) \; , \tag{54}
$$

where $F^{\tau 2}_2(z, Q^2)$ is given by Eqs. (16a–16f) at the LO approximation and by Eqs. (25a–25e) at NLO one, respectively. The twist-four term $F^{R\tau 4}_2(z, Q^2)$ has the form (3), i.e.

$$
\frac{1}{e} F^{R\tau 4}_2(z, Q^2) = \sum_{a=q,G} \mathrm{a}^{\tau 4}_a \, \widetilde{\mu}^{\tau 4}_a(z, Q^2) \otimes f^{\tau 2}_a(z, Q^2) \; ,
$$

where the symbol $\otimes$ marks the Mellin convolution

$$
A(z) \otimes B(z) = \int_z^1 \frac{dy}{y} \, A(y) \, B\left(\frac{z}{y}\right) \; . \tag{55}
$$

The corresponding Mellin transforms of $\widetilde{\mu}^{\tau 4,6}_a(z, Q^2)$

$$
\mu^{\tau 4,6}_a(n, Q^2) = \int_0^1 dz \, z^{n-1} \, \widetilde{\mu}^{\tau 4,6}_a(z, Q^2) \tag{56}
$$

are presented in the Appendix A (see Eqs. (A2–A4) and (A7)).

Looking the n-space representations for renormalon power-like corrections given in Appendix A and applying the technique to transform the Mellin convolutions to standard products at small x (see [54, 55] and Appendix B) we can represent the Eq. (3) in the form

$$
\begin{aligned}
\frac{1}{e} F^{R\tau 4}_2(z, Q^2) = \frac{64 T_R f}{15 \beta_0^2} \Bigg[& \mathrm{a}^{\tau 4}_G \left\{ \widehat{\delta}^{-1} + \frac{101}{120} + \frac{1}{2} \ln\left(\frac{Q^2}{|\mathrm{a}^{\tau 4}_G|}\right) \right\} f^{\tau 2}_G(z, Q^2) \\
& + 2 C_F \mathrm{a}^{\tau 4}_q \left\{ \widehat{\delta}^{-2} + \frac{11}{120} \widehat{\delta}^{-1} - \frac{2291}{3600} + \frac{1}{2} \ln\left(\frac{Q^2}{|\mathrm{a}^{\tau 4}_q|}\right) \left(\widehat{\delta}^{-1} - \frac{139}{120}\right) \right\} f^{\tau 2}_q(z, Q^2) \Bigg] \; ,
\end{aligned} \tag{57}
$$

The operators $\widehat{\delta}^{-1}$ and $\widehat{\delta}^{-2}$ are defined as follows (see Appendix B for details)

$$
\widehat{\delta}^{-1} \left[f^{\tau 2,-}_a(z, Q^2) \right] = \frac{1}{\delta_R} f^{\tau 2,-}_a(z, Q^2) \; , \quad \widehat{\delta}^{-2} \left[f^{\tau 2,-}_a(z, Q^2) \right] = \frac{1}{\delta_R^2} f^{\tau 2,-}_a(z, Q^2) \tag{58a}
$$

$$
\widehat{\delta}^{-1} \left[\rho^k \widetilde{I}_k(\sigma) \right] = \rho^{k-1} \, \widetilde{I}_{|k-1|}(\sigma) \quad , \quad \widehat{\delta}^{-2} \left[\rho^k \widetilde{I}_k(\sigma) \right] = \rho^{k-2} \, \widetilde{I}_{|k-2|}(\sigma) \; . \tag{58b}
$$

Note that the Eqs. (16) and Eqs. (25) have been obtained in [44] with the accuracy $\mathcal{O}(\rho)$ for the $'+'$ component and with one $\mathcal{O}(z)$ for the $'-'$ component, respectively. It leads to the fact that we should use only the most singular terms in the r.h.s. of Eq. (57): i.e. the terms $\widehat{\delta}^{-1}$ and $\sim \ln(Q^2/|\mathrm{a}_G^{\tau 4}|)$ for the gluon part and the terms $\widehat{\delta}^{-2}$ and $\ln(Q^2/|\mathrm{a}_q^{\tau 4}|)\,\widehat{\delta}^{-1}$ for the quark part.

Then, the Eq. (57) should be replaced by

$$\frac{1}{e}\,F_2^{R\tau 4}(z,Q^2) = \frac{64T_R f}{15\beta_0^2}\Bigg[\mathrm{a}_G^{\tau 4}\left\{\widehat{\delta}^{-1} + \frac{1}{2}\ln\left(\frac{Q^2}{|\mathrm{a}_G^{\tau 4}|}\right)\right\} f_G^{\tau 2}(z,Q^2) + 2C_F\mathrm{a}_q^{\tau 4}\left\{\widehat{\delta}^{-2} + \frac{1}{2}\ln\left(\frac{Q^2}{|\mathrm{a}_q^{\tau 4}|}\right)\widehat{\delta}^{-1}\right\} f_q^{\tau 2}(z,Q^2)\Bigg]\,, \tag{59}$$

Applying the operators $\widehat{\delta}^{-1}$ and $\widehat{\delta}^{-2}$ separately to the $'+'$ and $'-'$ components of $f_a^{\tau 2}(z,Q^2)$, we obtain the following results for $F_2^{R\tau 4}(z,Q^2)$:

$$F_2^{R\tau 4}(z,Q^2) \;=\; F_2^{R\tau 4,+}(z,Q^2) \;+\; F_2^{R\tau 4,-}(z,Q^2)\,, \tag{60a}$$

where

$$\begin{aligned}\frac{1}{e}\,F_2^{R\tau 4,+}(z,Q^2) &= \frac{32T_R f}{15\beta_0^2}\,f_G^{\tau 2,+}(z,Q^2)\Bigg[\mathrm{a}_G^{\tau 4}\left\{\frac{2}{\rho}\frac{\widetilde{I}_1(\sigma)}{\widetilde{I}_0(\sigma)} + \ln\left(\frac{Q^2}{|\mathrm{a}_G^{\tau 4}|}\right)\right\}\\ &+ \frac{4C_F T_R f}{3C_A}\mathrm{a}_q^{\tau 4}\Bigg(\left(1-\bar{d}_{+-}^q(1)a_s(Q^2)\right)\left\{\frac{2}{\rho}\frac{\widetilde{I}_1(\sigma)}{\widetilde{I}_0(\sigma)} + \ln\left(\frac{Q^2}{|\mathrm{a}_q^{\tau 4}|}\right)\right\}\\ &+ \frac{20C_A}{3}a_s(Q^2)\left\{\frac{2}{\rho^2}\frac{\widetilde{I}_2(\sigma)}{\widetilde{I}_0(\sigma)} + \ln\left(\frac{Q^2}{|\mathrm{a}_q^{\tau 4}|}\right)\frac{\widetilde{I}_1(\sigma)}{\rho\widetilde{I}_0(\sigma)}\right\}\Bigg)\Bigg]\,, \end{aligned} \tag{60b}$$

$$\begin{aligned}\frac{1}{e}\,F_2^{R\tau 4,-}(z,Q^2) &= \frac{32T_R f}{15\beta_0^2}\,f_G^{\tau 2,-}(z,Q^2)\Bigg[\mathrm{a}_G^{\tau 4}\ln\left(\frac{Q^2}{z_G^2\,|\mathrm{a}_G^{\tau 4}|}\right)\\ &- 2C_A\mathrm{a}_q^{\tau 4}\left\{\ln\left(\frac{1}{z_q}\right)\ln\left(\frac{Q^2}{z_q\,|\mathrm{a}_q^{\tau 4}|}\right) - p'(\nu_q)\right\}\Bigg]\,. \end{aligned} \tag{60c}$$

C. Incorporation of twist-six contributions in the framework of the renormalon model

We shortly demonstrate the twist-six contributions in the framework of the renormalon model.

When we added the twist-six part, the full small x asymptotic results for PD and $F_2^{\rm ren}$ structure function at NLO of perturbation theory:

$$F_2^R(x,Q^2) \;=\; F_2^{\tau 2}(x,Q^2) + \frac{1}{Q^2}\,F_2^{R\tau 4}(z,Q^2) \;+\; \frac{1}{Q^4}\,F_2^{R\tau 6}(z,Q^2)\,, \tag{61}$$

By analogy with twist-four case the twist-six term $f_q^{R\tau 6}(z,Q^2)$ has the form:

$$\frac{1}{e}\,F_2^{R\tau 6}(z,Q^2) \;=\; \sum_{a=q,G}\mathrm{a}_a^{\tau 6}\,\widetilde{\mu}_a^{\tau 6}(z,Q^2)\otimes f_a^{\tau 2}(z,Q^2)\,, \tag{62}$$

where $\widetilde{\mu}_a^{\tau 6}(z,Q^2)$ are given in [50]. The corresponding Mellin transform of $\mu_a^{\tau 6}(n,Q^2)$ is presented in the Appendix A (see Eqs. (A2), (A5,A6) and (A7)).

By analogy with the previous subsection applying the technique to transform the Mellin convolutions to the standard products at small x (see [54, 55] and Appendix B), we can represent the Eq. (62) in the form

$$\begin{aligned}\frac{1}{e}\,F_2^{R\tau 6}(z,Q^2) = -\frac{8}{7}\times\frac{64T_R f}{15\beta_0^2}\Bigg[&\mathrm{a}_G^{\tau 6}\left\{\widehat{\delta}^{-1} + \frac{2663}{3360} + \frac{1}{2}\ln\left(\frac{Q^2}{\sqrt{|\mathrm{a}_G^{\tau 6}|}}\right)\right\} f_G^{\tau 2}(z,Q^2)\\ &+ 2C_F\mathrm{a}_q^{\tau 6}\left\{\widehat{\delta}^{-2} + \frac{143}{3360}\widehat{\delta}^{-1} - \frac{870637}{1411200} + \frac{1}{2}\ln\left(\frac{Q^2}{\sqrt{|\mathrm{a}_q^{\tau 6}|}}\right)\left(\widehat{\delta}^{-1} - \frac{3217}{3360}\right)\right\} f_q^{\tau 2}(z,Q^2)\Bigg]\end{aligned} \tag{63}$$

Considering only the most singular terms in the r.h.s. of (63), i.e. the terms $\widehat{\delta}^{-1}$ and $\sim \ln(Q^2/\sqrt{|\mathrm{a}_G^{\tau 6}|})$ for the gluon part and the terms $\widehat{\delta}^{-2}$ and $\ln(Q^2/\sqrt{|\mathrm{a}_q^{\tau 6}|})\,\widehat{\delta}^{-1}$ for the quark part, we obtain immediately the following results:

$$\frac{1}{e}\,F_2^{R\tau 6}(z,Q^2) = -\frac{8}{7}\times\frac{64T_Rf}{15\beta_0^2}\Biggl[\mathrm{a}_G^{\tau 6}\left\{\widehat{\delta}^{-1}+\frac{1}{2}\ln\left(\frac{Q^2}{\sqrt{|\mathrm{a}_G^{\tau 6}|}}\right)\right\}f_G^{\tau 2}(z,Q^2) + 2C_F\mathrm{a}_q^{\tau 6}\left\{\widehat{\delta}^{-2}+\frac{1}{2}\ln\left(\frac{Q^2}{\sqrt{|\mathrm{a}_q^{\tau 6}|}}\right)\widehat{\delta}^{-1}\right\}f_q^{\tau 2}(z,Q^2)\Biggr]\,, \tag{64}$$

which is very close to the twist-four one, see Eq. (59):

$$\frac{1}{e}\,F_2^{R\tau 6}(z,Q^2) = -\frac{8}{7}\times\left[f_q^{R\tau 4}(z,Q^2)\ \text{ with }\ \mathrm{a}_a^{\tau 4}\to\mathrm{a}_a^{\tau 6}\,,\ \ln\left(\frac{Q^2}{|\mathrm{a}_a^{\tau 4}|}\right)\to\ln\left(\frac{Q^2}{\sqrt{|\mathrm{a}_a^{\tau 6}|}}\right)\right]\,. \tag{65}$$

Note that the representation (65) of the twist-six terms in the terms of the twist-four ones is universal and has quite compact form and, thus, it will be often used below.

Because the forms of the twist-four and twist-six contributions are very similar, it is possible to present quite compact form for the full contribution of the higher-twist operators $F_2^{Rh\tau}(z,Q^2)$

$$F_2^R(z,Q^2) \;=\; F_2^{\tau 2}(z,Q^2) \;+\; F_2^{Rh\tau}(z,Q^2)\,, \tag{66a}$$

where

$$F_2^{Rh\tau}(z,Q^2) \;=\; F_2^{Rh\tau,+}(z,Q^2) \;+\; F_2^{Rh\tau,-}(z,Q^2) \tag{66b}$$

and

$$\begin{aligned}\frac{1}{e}\,F_2^{Rh\tau,+}(z,Q^2) \;&=\frac{32T_Rf}{15\beta_0^2}\,f_G^{\tau 2,+}(z,Q^2)\sum_{m=4,6}k_m\Biggl[\frac{\mathrm{a}_G^{\tau m}}{Q^{(m-2)}}\left\{\frac{2}{\rho}\frac{\widetilde{I}_1(\sigma)}{\widetilde{I}_0(\sigma)}+\ln\left(\frac{Q^2}{|\mathrm{a}_G^{\tau m}|^{p_m}}\right)\right\}\\ &+\frac{4C_FT_Rf}{3C_A}\frac{\mathrm{a}_q^{\tau m}}{Q^{(m-2)}}\Biggl(\left(1-\bar{d}_{+-}^q(1)a_s(Q^2)\right)\left\{\frac{2}{\rho}\frac{\widetilde{I}_1(\sigma)}{\widetilde{I}_0(\sigma)}+\ln\left(\frac{Q^2}{|\mathrm{a}_q^{\tau m}|^{p_m}}\right)\right\}\\ &+\frac{20C_A}{3}a_s(Q^2)\left\{\frac{2}{\rho^2}\frac{\widetilde{I}_2(\sigma)}{\widetilde{I}_0(\sigma)}+\ln\left(\frac{Q^2}{|\mathrm{a}_q^{\tau m}|^{p_m}}\right)\frac{\widetilde{I}_1(\sigma)}{\rho\widetilde{I}_0(\sigma)}\right\}\Biggr)\Biggr]\,,\end{aligned} \tag{66c}$$

$$\begin{aligned}\frac{1}{e}\,F_2^{Rh\tau,-}(z,Q^2) \;&=\frac{32T_Rf}{15\beta_0^2}\,f_G^{\tau 2,-}(z,Q^2)\sum_{m=4,6}k_m\Biggl[\frac{\mathrm{a}_G^{\tau m}}{Q^{(m-2)}}\ln\left(\frac{Q^2}{z_G^2\,|\mathrm{a}_G^{\tau m}|^{p_m}}\right)\\ &-2C_A\frac{\mathrm{a}_q^{\tau m}}{Q^{(m-2)}}\left\{\ln\left(\frac{1}{z_q}\right)\ln\left(\frac{Q^2}{z_q\,|\mathrm{a}_q^{\tau m}|^{p_m}}\right)-p'(\nu_q)\right\}\Biggr]\,,\end{aligned} \tag{66d}$$

where $k_4=1$, $k_6=-8/7$ and $p_4=1$, $p_6=1/2$.

VI. THE HIGHER TWIST CONTRIBUTIONS FOR THE DERIVATIVE $\partial F_2/\partial\ln Q^2$

By analogy with the previous Section we consider firstly only the twist-four terms in the framework of the infrared renormalon model. The contribution of the twist-six terms will be incorporated shortly at the end of this Section.

A. Renormalon model predictions for twist-four operators

Note that there are the following properties

$$\frac{d}{d\ln Q^2}\frac{1}{Q^2} \;=\; -\frac{1}{Q^2}\,,\quad \frac{d}{d\ln Q^2}\left[\frac{1}{Q^2}\ln\left(\frac{\Lambda^2}{Q^2}\right)\right] \;=\; -\frac{1}{Q^2}\left(\ln\left(\frac{\Lambda^2}{Q^2}\right)+1\right) \;\approx\; -\frac{1}{Q^2}\ln\left(\frac{\Lambda^2}{Q^2}\right)\,, \tag{67}$$

where we keep only most important terms (see discussions in the previous Section and Eq. (59)).

In this approximation we easily obtain that

$$\frac{\partial F_2^{R}(z,Q^2)}{\partial \ln Q^2} \;=\; \frac{\partial F_2^{\tau 2}(z,Q^2)}{\partial \ln Q^2} + \frac{1}{Q^2}\left(\frac{\partial F_2^{R\tau 4}(z,Q^2)}{\partial \ln Q^2} - F_2^{R\tau 4}(z,Q^2)\right) \tag{68a}$$

and

$$\frac{\partial F_2^{R\tau 4}(z,Q^2)}{\partial \ln Q^2} \;=\; e\,\frac{8T_R f}{3}\, a_s(Q^2)\,\Phi_G^{R\tau 4}(z,Q^2)\;. \tag{68b}$$

The value of $F_2^{R\tau 4}(z,Q^2)$ is given by Eqs. (60a)–(60c) and

$$\begin{aligned}\Phi_G^{R\tau 4}(z,Q^2) \;=&\frac{16C_A}{5\beta_0^2} f_G^{\tau 2,+}(z,Q^2)\left[\mathrm{a}_G^{\tau 4}\left\{\frac{2}{\rho^2}\frac{\widetilde{I}_2(\sigma)}{\widetilde{I}_0(\sigma)} + \ln\left(\frac{Q^2}{|\mathrm{a}_G^{\tau 4}|}\right)\frac{1}{\rho}\frac{\widetilde{I}_1(\sigma)}{\widetilde{I}_0(\sigma)}\right\}\right.\\ &\left. + \frac{4C_F T_R f}{3C_A}\mathrm{a}_q^{\tau 4}\left\{\frac{2}{\rho^2}\frac{\widetilde{I}_2(\sigma)}{\widetilde{I}_0(\sigma)} + \ln\left(\frac{Q^2}{|\mathrm{a}_q^{\tau 4}|}\right)\frac{1}{\rho}\frac{\widetilde{I}_1(\sigma)}{\widetilde{I}_0(\sigma)}\right\}\right]\;.\end{aligned} \tag{68c}$$

Thus, we see that the twist-four corrections to F_2 and $dF_2/d\ln Q^2$ have opposite signs, because $dF_2^{R\tau 4}/d\ln Q^2 \sim a_s(Q^2)$ and the most important twist-four contribution is given by $F_2^{R\tau 4}(z,Q^2)$.

B. Incorporation of twist-six contributions in the framework of the renormalon model

Following to the subsection V C of the previous Section and considering the properties

$$\frac{d}{d\ln Q^2}\frac{1}{Q^4} \;=\; -\frac{2}{Q^4}\;,\quad \frac{d}{d\ln Q^2}\left[\frac{1}{Q^4}\ln\left(\frac{\Lambda^2}{Q^2}\right)\right] \;=\; -\frac{1}{Q^2}\left(2\ln\left(\frac{\Lambda^2}{Q^2}\right)+1\right) \;\approx\; -\frac{2}{Q^2}\ln\left(\frac{\Lambda^2}{Q^2}\right)\;, \tag{69}$$

together with the one (29), we immediately obtain that

$$\frac{\partial F_2^{R}(z,Q^2)}{\partial \ln Q^2} \;=\; \frac{\partial F_2^{\tau 2}(z,Q^2)}{\partial \ln Q^2} + \frac{1}{Q^2}\left(\frac{\partial F_2^{R\tau 4}(z,Q^2)}{\partial \ln Q^2} - F_2^{R\tau 4}(z,Q^2)\right) + \frac{1}{Q^4}\left(\frac{\partial F_2^{R\tau 6}(z,Q^2)}{\partial \ln Q^2} - 2F_2^{R\tau 6}(z,Q^2)\right) \tag{70a}$$

and

$$\frac{\partial F_2^{R\tau 6}(z,Q^2)}{\partial \ln Q^2} \;=\; e\,\frac{8T_R f}{3}\, a_s(Q^2)\,\Phi_G^{R\tau 6}(z,Q^2)\;. \tag{70b}$$

The value of $f_q^{R\tau 6}(z,Q^2)$ is given by Eq. (65) and

$$\Phi_G^{R\tau 6}(z,Q^2) \;=\; -\frac{8}{7}\times\left[\Phi_G^{R\tau 4}(z,Q^2)\;\;\text{with}\;\;\mathrm{a}_a^{\tau 4}\to\mathrm{a}_a^{\tau 6},\;\ln\left(\frac{Q^2}{|\mathrm{a}_a^{\tau 4}|}\right)\to\ln\left(\frac{Q^2}{\sqrt{|\mathrm{a}_a^{\tau 6}|}}\right)\right]\;. \tag{70c}$$

Thus, we see that by analogy with the case of $F_2(z,Q^2)$ itself, for the derivation (70) the twist-six terms partially compensate the contributions of the twist-four terms.

VII. PARTON DISTRIBUTION FUNCTIONS IN THE RENORMALON MODEL APPROACH

It is clearly to see that the standard parton distributions $f_q(x,Q^2)$ and $f_G(x,Q^2)$ fitted with help of experimental data do not coincide with the above twist-two ones $f_q^{\tau 2}(z,Q^2)$ and $f_G^{\tau 2}(z,Q^2)$. These PD $f_q(z,Q^2)$ and $f_G(z,Q^2)$ are usually defined keeping their twist-two relations (16a) or (25a) with the structure function $F_2(z,Q^2)$, i.e.

At LO

$$F_2(z,Q^2) \;=\; e\,f_q(z,Q^2)\;, \tag{71}$$

At NLO

$$F_2(z,Q^2) \;=\; e\,\left(f_q(z,Q^2) \,+\, \frac{8T_Rf}{3}\, a_s(Q^2)\, f_G(z,Q^2) \right)\;. \tag{72}$$

Thus, the parton distributions $f_q(z,Q^2)$ and $f_G(z,Q^2)$ can be strongly deviated for the corresponding the twist-two densities $f_q^{\tau 2}(z,Q^2)$ and $f_G^{\tau 2}(z,Q^2)$ at quite low Q^2 values, because there are the HT corrections to $F_2^{\tau 2}(z,Q^2)$.

The HT correction to the parton dstribution at the LO was presented in the Introduction already. Here we present the results at the NLO. As it was in the previous Section, we consider firstly the twist-four corrections.

A. Twist-four corrections to (singlet) quark distribution

Consider firstly the (singlet) quark parton distribution $f_q(z,Q^2)$. From the Eq. (16a) and the analysis of the Section V we can obtain that

$$f_q^R(z,Q^2) \;=\; f_q^{\tau 2}(z,Q^2) \,+\, \frac{1}{Q^2}\, f_q^{R\tau 4}(z,Q^2)\;, \tag{73a}$$

where $f_q^{R\tau 4}(z,Q^2)$ is given at the LO by Eqs. (12a) and (12b).

It is useful to represent also the complete expressions directly for $f_q^R(z,Q^2)$:

$$f_q^R(z,Q^2) \;=\; f_q^{R,+}(z,Q^2) \,+\, f_q^{R,-}(z,Q^2)\;, \tag{73b}$$

where at the NLO

$$\frac{f_q^{R,+}(z,Q^2)}{f_q^{\tau 2,+}(z,Q^2)} \;=\; 1 + \frac{64C_FT_Rf}{15\beta_0^2}\,\frac{\mathrm{a}_q^{\tau 4}}{Q^2}\Biggl\{ \frac{2}{\rho^2}\,\frac{\widetilde{I}_1(\sigma)\left(1-\bar{d}^q_{+-}(1)a_s(Q^2)\right) + (20C_A/3)a_s(Q^2)\widetilde{I}_2(\sigma)/\rho}{\widetilde{I}_1(\sigma)\left(1-\bar{d}^q_{+-}(1)a_s(Q^2)\right) + (20C_A/3)a_s(Q^2)\widetilde{I}_0(\sigma)/\rho}$$
$$+ \ln\left(\frac{Q^2}{\left|\mathrm{a}_q^{\tau 4}\right|}\right)\frac{1}{\rho}\,\frac{\widetilde{I}_0(\sigma)\left(1-\bar{d}^q_{+-}(1)a_s(Q^2)\right) + (20C_A/3)a_s(Q^2)\widetilde{I}_1(\sigma)/\rho}{\widetilde{I}_1(\sigma)\left(1-\bar{d}^q_{+-}(1)a_s(Q^2)\right) + (20C_A/3)a_s(Q^2)\widetilde{I}_0(\sigma)/\rho}\Biggr\} \,+\, \mathcal{O}(\rho)\;, \tag{73c}$$

$$\frac{f_q^{R,-}(z,Q^2)}{f_q^{\tau 2,-}(z,Q^2)} \;=\; 1 + \frac{64C_FT_Rf}{15\beta_0^2}\,\frac{\mathrm{a}_q^{\tau 4}}{Q^2}\left\{ \ln\left(\frac{1}{z_q}\right)\ln\left(\frac{Q^2}{z_q\left|\mathrm{a}_q^{\tau 4}\right|}\right) - p'(\nu_q)\right\} \,+\, \mathcal{O}(z)\;. \tag{73d}$$

We clearly see that the twist-four terms are responsible for the additional positive contributions to the quark distribution, which are very important at low Q^2 values.

So, the experimentally extracted quark distribution $f_q(z,Q^2)$, which have the leading twist relations (71) and (72) with $F_2(z,Q^2)$, strongly deviates form the leading twist quark distribution $f_q^{\tau 2}(z,Q^2)$. At quite low Q^2 values, where $f_q^{\tau 2}(z,Q^2)$ had the quite flat behavior closed to (15), the full quark distribution $f_q^R(z,Q^2)$ will rise at $z\to 0$ (see Eqs. (73c) and (73d)), because $\mathrm{a}_q^{\tau 4}>0$ (see Tables VI ,VII). This rise is in full agreement with the corresponding experimental data (see Tables VI ,VII, Figure 9, Section X and discussions therein).

B. Twist-four corrections to gluon distribution

Consider now the gluon parton distribution $f_G(z,Q^2)$. From the Eq. (16a) and the analysis of the Section V we can obtain that

$$f_G^R(z,Q^2) \;=\; f_G^{\tau 2}(z,Q^2) \,+\, \frac{1}{Q^2}\, f_G^{R\tau 4}(z,Q^2)\;. \tag{74a}$$

where $f_q^{R\tau 4}(z,Q^2)$ is given at the LO by Eqs. (12c) and (12d).

For the gluon distribution in the NLO we have the similar relations

$$f_G^R(z,Q^2) \;=\; f_G^{R,+}(z,Q^2) \,+\, f_G^{R,-}(z,Q^2)\;, \tag{74b}$$

$$\frac{f_G^{R,+}(z,Q^2)}{f_G^{\tau 2,+}(z,Q^2)} \;=\; 1 + \frac{8}{5\beta_0^2}\,\frac{\mathrm{a}_G^{\tau 4}}{a_s(Q^2)\,Q^2}\left\{ \frac{2}{\rho}\,\frac{\widetilde{I}_1(\sigma)}{\widetilde{I}_0(\sigma)} + \ln\left(\frac{Q^2}{\left|\mathrm{a}_G^{\tau 4}\right|}\right)\right\} \,+\, \mathcal{O}(\rho)\;, \tag{74c}$$

$$\frac{f_G^{R,-}(z,Q^2)}{f_G^{\tau 2,-}(z,Q^2)} \;=\; 1 + \frac{8}{5\beta_0^2}\,\frac{\mathrm{a}_G^{\tau 4}}{a_s(Q^2)\,Q^2}\,\ln\left(\frac{Q^2}{z_G^2\left|\mathrm{a}_G^{\tau 4}\right|}\right) \,+\, \mathcal{O}(z)\;. \tag{74d}$$

So, as in the case of the quark distribution, the experimentally extracted gluon density $f_G(z,Q^2)$, which has the leading twist relation with $F_2(z,Q^2)$ and $dF_2/d\ln Q^2$, strongly deviates form the leading twist gluon distribution $f_G^{\tau 2}(z,Q^2)$. At quite low Q^2 values: $Q^2 \sim Q_0^2$, where $f_G^{\tau 2}(z,Q^2)$ had the quite flat behavior closed to (15), the full gluon distribution $f_q^R(z,Q^2)$ falls at $x \to 0$, because $\mathrm{a}_G^{\tau 4} < 0$ (see Tables VI ,VII). The behavior is in full agreement with the corresponding experimental data (see Tables VI ,VII, Figure 9, Section X and discussions therein).

C. Twist-six corrections to parton distributions

We shortly demonstrate the twist-six contributions to parton distribution in the framework of the renormalon model. When we added the twist-six part, the full small x asymptotic results for parton distributions is

$$f_a(z,Q^2) \;=\; f_a^{\tau 2}(z,Q^2) \;+\; \frac{1}{Q^2}\, f_a^{R\tau 4}(z,Q^2) \;+\; \frac{1}{Q^4}\, f_a^{R\tau 6}(z,Q^2) \;=\; f_a^{\tau 2}(z,Q^2) \;+\; f_a^{Rh\tau}(z,Q^2)\;, \tag{75}$$

where $f_a^{R\tau 6}(z,Q^2)$ are given by Eqs. (14):

$$f_a^{R\tau 6}(z,Q^2) \;=\; -\frac{8}{7}\times\left[f_a^{R\tau 4}(z,Q^2)\;\text{ with }\; \mathrm{a}_a^{\tau 4}\to \mathrm{a}_a^{\tau 6}\,,\; \ln\left(\frac{Q^2}{|\mathrm{a}_a^{\tau 4}|}\right)\to \ln\left(\frac{Q^2}{\sqrt{|\mathrm{a}_a^{\tau 6}|}}\right)\right]\;,$$

The twist-six corrections do not change the results for parton distributions obtained in the previous subsection.

VIII. THE HIGHER TWIST CONTRIBUTIONS TO THE SLOPES OF F_2 AND OF PARTON DISTRIBUTIONS

Consider the power-like corrections to the twist-two effective slopes $\lambda_{F2}^{\mathrm{eff},\tau 2}(z,Q^2)$ and $\lambda_a^{\mathrm{eff},\tau 2}(z,Q^2)$ $(a = q,G)$ introduced in the Section IV. The effective slopes have the following form

$$\lambda_{F2}^{\mathrm{eff}}(z,Q^2) \;=\; \frac{\partial}{\partial\ln(1/z)}\ln\left[F_2^{\tau 2}(z,Q^2) \;+\; \frac{1}{Q^2}\,F_2^{R\tau 4}(z,Q^2) \;+\; \frac{1}{Q^4}\,F_2^{R\tau 6}(z,Q^2)\right]\;, \tag{76}$$

$$\lambda_a^{\mathrm{eff}}(z,Q^2) \;=\; \frac{\partial}{\partial\ln(1/z)}\ln\left[f_a^{\tau 2}(z,Q^2) \;+\; \frac{1}{Q^2}\,f_a^{R\tau 4}(z,Q^2) \;+\; \frac{1}{Q^4}\,f_a^{R\tau 6}(z,Q^2)\right]\;. \tag{77}$$

Using Eqs. (38), the derivations $\partial F_2^{\tau 2}/\partial\ln(1/z)$, $\partial f_a^{\tau 2}/\partial\ln(1/z)$ and $\partial f_a^{R\tau m}/\partial\ln(1/z)$, $(m = 4,6)$, can be represented as the sum of two components ('+' and '−') which are obtained from the corresponding ('+' and '−') PD functions. One can show that

$$\frac{\partial f_q^{R\tau 4,+}(z,Q^2)}{\partial\ln(1/z)} \;=\; \frac{64C_FT_Rf}{15\beta_0^2}\,\mathrm{a}_q^{\tau 4}\left\{\frac{2}{\rho}\,\frac{\widetilde{I}_0(\sigma)\left(1-\bar{d}_{+-}^q(1)a_s(Q^2)\right)+(20C_A/3)a_s(Q^2)\widetilde{I}_1(\sigma)/\rho}{\widetilde{I}_1(\sigma)\left(1-\bar{d}_{+-}^q(1)a_s(Q^2)\right)+(20C_A/3)a_s(Q^2)\widetilde{I}_0(\sigma)/\rho}\right.$$
$$\left. +\ln\left(\frac{Q^2}{|\mathrm{a}_q^{\tau 4}|}\right)\right\} f_q^{\tau 2,+}(z,Q^2)\;, \tag{78a}$$

$$\frac{\partial f_q^{R\tau 4,-}(z,Q^2)}{\partial\ln(1/z)} \;=\; \frac{64C_FT_Rf}{15\beta_0^2}\,\mathrm{a}_q^{\tau 4}\ln\left(\frac{Q^2}{z_q^2\,|\mathrm{a}_q^{\tau 4}|}\right) f_q^{\tau 2,-}(z,Q^2)\;, \tag{78b}$$

$$\frac{\partial f_G^{R\tau 4,+}(z,Q^2)}{\partial\ln(1/z)} \;=\; \frac{8}{5\beta_0^2}\,\frac{\mathrm{a}_G^{\tau 4}}{a_s(Q^2)}\left\{2+\ln\left(\frac{Q^2}{|\mathrm{a}_G^{\tau 4}|}\right)\,\rho\,\frac{\widetilde{I}_1(\sigma)}{\widetilde{I}_0(\sigma)}\right\} f_G^{\tau 2,+}(z,Q^2)\;, \tag{78c}$$

$$\frac{\partial f_G^{R\tau 4,-}(z,Q^2)}{\partial\ln(1/z)} \;=\; \frac{16}{5\beta_0^2}\,\frac{\mathrm{a}_G^{\tau 4}}{a_s(Q^2)}\,f_G^{\tau 2,-}(z,Q^2)\;, \tag{78d}$$

$$\frac{\partial f_a^{R\tau 6,\pm}(z,Q^2)}{\partial\ln(1/z)} \;=\; -\frac{8}{7}\times\left[\frac{\partial f_a^{R\tau 4,\pm}(z,Q^2)}{\partial\ln(1/z)}\;\text{ with }\; \mathrm{a}_a^{\tau 4}\to\mathrm{a}_a^{\tau 6}\;,\quad \ln\left(\frac{Q^2}{|\mathrm{a}_a^{\tau 4}|}\right)\to\ln\left(\frac{Q^2}{\sqrt{|\mathrm{a}_a^{\tau 6}|}}\right)\right]\;. \tag{78e}$$

The Eqs. (76) and (77) together with the Eqs. (42) and (78) give a complete information about the full and asymptotical values of the slopes $\lambda_{F2}^{\mathrm{eff}}(z,Q^2)$ and $\lambda_a^{\mathrm{eff}}(z,Q^2)$. The results will be demonstrated on Figs. 2, 6 and 7.

It is possible, however, to give a simple demonstration of the effect of the HT corrections. Following the Section IV, we can prepare also the results for the higher twist corrections to the asymptotical values of $\lambda^{\rm eff,\tau2}_{F2}(z,Q^2)$ and $\lambda^{\rm eff,\tau2}_{a}(z,Q^2)$, which can be obtained by neglecting the $'-'$ components. Restricting ourselves by the twist-four case we can estimate the value of the slopes $\lambda^{\rm eff}_{F2,\rm as}(z,Q^2)$ and $\lambda^{\rm eff}_{a,\rm as}(z,Q^2)$ in the form

$$\lambda^{\rm eff}_{F2,\rm as}(z,Q^2) \;=\; \lambda^{\rm eff,\tau2}_{F2,\rm as}(z,Q^2) + \frac{1}{Q^2}\,\lambda^{{\rm eff},R\tau4}_{F2,\rm as}(z,Q^2) + \mathcal{O}\left(\frac{1}{Q^4}\right)\,, \tag{79}$$

$$\lambda^{\rm eff}_{a,\rm as}(z,Q^2) \;=\; \lambda^{\rm eff,\tau2}_{a,\rm as}(z,Q^2) + \frac{1}{Q^2}\lambda^{{\rm eff},R\tau4}_{a,\rm as}(z,Q^2) + \mathcal{O}\left(\frac{1}{Q^4}\right)\,, \tag{80}$$

where at the LO

$$\lambda^{{\rm eff},R\tau4}_{F2,\rm as}(z,Q^2) \;=\; \frac{16C_A}{5\beta_0^2}\Biggl[{\rm a}^{\tau4}_G\Biggl\{2\frac{\widetilde{I}_0(\sigma_{\rm LO})-\widetilde{I}_2(\sigma_{\rm LO})}{\rho_{\rm LO}\widetilde{I}_1(\sigma_{\rm LO})} + \ln\left(\frac{Q^2}{{\rm a}^{\tau4}_G}\right)\left(1-\frac{\widetilde{I}_0(\sigma_{\rm LO})\widetilde{I}_2(\sigma_{\rm LO})}{\widetilde{I}^2_1(\sigma_{\rm LO})}\right)\Biggr\}$$

$$+\;\frac{4C_FT_Rf}{3C_A}{\rm a}^{\tau4}_q\Biggl\{2\frac{\widetilde{I}_0(\sigma_{\rm LO})-\widetilde{I}_2(\sigma_{\rm LO})}{\rho_{\rm LO}\widetilde{I}_1(\sigma_{\rm LO})} + \ln\left(\frac{Q^2}{{\rm a}^{\tau4}_G}\right)\left(1-\frac{\widetilde{I}_0(\sigma_{\rm LO})\widetilde{I}_2(\sigma_{\rm LO})}{\widetilde{I}^2_1(\sigma_{\rm LO})}\right)\Biggr\}\Biggr] \tag{81a}$$

$$\approx\; \frac{16C_A}{5\beta_0^2}\frac{1}{2\rho_{\rm LO}\ln(1/z)}\Biggl[{\rm a}^{\tau4}_G\Biggl\{\frac{4}{\rho_{\rm LO}} + \ln\left(\frac{Q^2}{{\rm a}^{\tau4}_G}\right)\Biggr\} + \frac{4C_FT_Rf}{3C_A}{\rm a}^{\tau4}_q\Biggl\{\frac{4}{\rho_{\rm LO}} + \ln\left(\frac{Q^2}{{\rm a}^{\tau4}_q}\right)\Biggr\}\Biggr] \tag{81b}$$

$$\lambda^{{\rm eff},R\tau4}_{q,\rm as}(z,Q^2) \;=\; \frac{64C_FT_Rf}{15\beta_0^2}\,{\rm a}^{\tau4}_q\Biggl\{2\frac{\widetilde{I}_0(\sigma_{\rm LO})-\widetilde{I}_2(\sigma_{\rm LO})}{\rho\widetilde{I}_1(\sigma_{\rm LO})} + \ln\left(\frac{Q^2}{{\rm a}^{\tau4}_q}\right)\left(1-\frac{\widetilde{I}_0(\sigma_{\rm LO})\widetilde{I}_2(\sigma_{\rm LO})}{\widetilde{I}^2_1(\sigma_{\rm LO})}\right)\Biggr\} \tag{81c}$$

$$\approx\; \frac{64C_FT_Rf}{15\beta_0^2}\frac{{\rm a}^{\tau4}_q}{2\rho_{\rm LO}\ln(1/z)}\Biggl\{\frac{4}{\rho_{\rm LO}} + \ln\left(\frac{Q^2}{{\rm a}^{\tau4}_q}\right)\Biggr\} \tag{81d}$$

$$\lambda^{{\rm eff},R\tau4}_{G,\rm as}(z,Q^2) \;=\; \frac{16}{5\beta_0^2}\,\frac{{\rm a}^{\tau4}_G}{a_s(Q^2)}\left(1-\frac{\widetilde{I}^2_1(\sigma_{\rm LO})}{\widetilde{I}^2_0(\sigma_{\rm LO})}\right) \;\approx\; \frac{8}{5\beta_0^2}\,\frac{{\rm a}^{\tau4}_G}{a_s(Q^2)}\,\frac{1}{\rho_{\rm LO}\ln(1/z)}\;. \tag{81e}$$

From the equations (81b) and (81d) it possible to see that the slopes $\lambda^{\rm eff}_{F2,\rm as}(z,Q^2)$ and $\lambda^{\rm eff}_{q,\rm as}(z,Q^2)$ have got the positive twist-four corrections, that is in full agreement with the corresponding experimental H1 and ZEUS data for the slope λ_{F2} at low Q^2 values (see Fig. 7). However, the difference between the twist-four corrections to these slopes is negative , because ${\rm a}^{\tau4}_G < 0$ (see Tables 4, 6–8):

$$\lambda^{{\rm eff},R\tau4}_{F2,\rm as}(z,Q^2) \;-\; \lambda^{{\rm eff},R\tau4}_{q,\rm as}(z,Q^2) \;\approx\; \frac{8C_A}{5\beta_0^2}\,\frac{{\rm a}^{\tau4}_G}{\rho_{\rm LO}\ln(1/z)}\,\Biggl\{\frac{4}{\rho_{\rm LO}} + \ln\left(\frac{Q^2}{{\rm a}^{\tau4}_q}\right)\Biggr\}\;. \tag{82}$$

Thus, the inequality $\lambda^{\rm eff}_{F2,\rm as}(z,Q^2) > \lambda^{\rm eff}_{q,\rm as}(z,Q^2)$ coming form Eq. (44b) takes place for not very small Q^2, because it is suppressed by power corrections.

We would like to note that the equations (81) are valid only at not very small Q^2 values, where we can neglect the terms $\sim 1/Q^4$ coming from expanding the denominator and from the twist-six terms. The small Q^2 behavior of $\lambda^{\rm eff}_{a,\rm as}(z,Q^2)$ can be easy demonstrated at the point $Q^2=Q^2_0$ in the following Section.

IX. PARTON DISTRIBUTIONS IN THE RENORMALON MODEL AT Q^2_0

As it has been already shown in the previous Section the total PD functions $f_q(z,Q^2)$ and $f_G(z,Q^2)$ fitted in experiments data do not coincide with the above twist-two ones $f^{\tau2}_q(z,Q^2)$ and $f^{\tau2}_G(z,Q^2)$. It is very useful to demonstrate the difference at Q^2_0, at the starting point of the DGLAP evolution.

We begin the analysis with the consideration only the twist-four terms. The results can be calculated from the final formulae of the previous Section but it is simpler to repeat all calculations given in the Section V. At $Q^2=Q^2_0$ all results simplify essentially because the leading-twist parton distributions are constant A_q and A_G at the point.

A. Parton distributions at Q_0^2

From the Eqs. (15) and (73) we can easy obtain at $Q^2 = Q_0^2$, that

$$f_a(z,Q_0^2) \;=\; A_a^{\tau 2} \;+\; \frac{1}{Q_0^2}\, f_a^{R\tau 4}(z,Q_0^2) \;+\; \frac{1}{Q_0^4}\, f_a^{R\tau 6}(z,Q_0^2)\;, \tag{83a}$$

where at the LO

$$\begin{aligned} f_q^{R\tau 4}(z,Q_0^2) &= \frac{64 C_F T_R f}{15\beta_0^2}\, \mathrm{a}_q^{\tau 4} \Biggl[A_q^{\tau 2}\left\{ \ln\left(\frac{1}{z_q}\right) \ln\left(\frac{Q_0^2}{z_q\,|\mathrm{a}_q^{\tau 4}|}\right) - p'(\nu_q) \right\} \\ &\qquad + \frac{2T_R f}{3C_A}\left(A_G^{\tau 2} + \frac{C_F}{C_A}A_q^{\tau 2}\right) \ln\left(\frac{Q_0^2}{z^2\,|\mathrm{a}_q^{\tau 4}|}\right)\Biggr] \end{aligned} \tag{83b}$$

$$f_G^{R\tau 4}(z,Q_0^2) = \frac{8}{5\beta_0^2}\,\frac{\mathrm{a}_G^{\tau 4}}{a_s(Q_0^2)} \left[A_G^{\tau 2} \ln\left(\frac{Q_0^2}{z^2\,|\mathrm{a}_G^{\tau 4}|}\right) + 2\frac{C_F}{C_A}A_q^{\tau 2} p(\nu_G)\right]\,, \tag{83c}$$

$$f_a^{R\tau 6}(z,Q_0^2) \;=\; -\,\frac{8}{7}\times\left[f_a^{R\tau 4}(z,Q_0^2)\;\; \text{with} \;\; \mathrm{a}_a^{\tau 4} \to \mathrm{a}_a^{\tau 6}\,,\; \ln\left(\frac{Q^2}{|\mathrm{a}_a^{\tau 4}|}\right) \to \ln\left(\frac{Q^2}{\sqrt{|\mathrm{a}_a^{\tau 6}|}}\right)\right]\,. \tag{83d}$$

Thus, the total parton distributions $f_q(z,Q_0^2)$ and $f_G(z,Q_0^2)$ are strongly deviated for the corresponding the twist-two densities $f_q^{\tau 2}(z,Q_0^2) = A_q^{\tau 2}$ and $f_G^{\tau 2}(z,Q_0^2) = A_G^{\tau 2}$. Because usually the fitted values of $\mathrm{a}_q^{\tau 4}$ ($\mathrm{a}_G^{\tau 4}$) are positive (negative), the twist-four terms lead to positive and negative contributions in the case of quark and gluon densities, respectively. The twist-six terms do not change the results essentially.

B. The effective slopes of F_2 and of parton distributions at Q_0^2

To estimate the values of the effective slopes at low Q^2 values we can look on their behavior at Q_0^2, where our formulae simplifies essentially. In the approximation, when the twist six contributions are negligible, we can easy obtain from the Eqs. (83)

$$\lambda_q^{\mathrm{eff},R}(z,Q_0^2) \;=\; \frac{64 C_F T_R f}{15\beta_0^2}\,\frac{\mathrm{a}_q^{\tau 4}}{Q_0^2}\left\{ \ln\left(\frac{Q_0^2}{z_q^2\,|\mathrm{a}_G^{\tau 4}|}\right) \;+\; \frac{4T_R f}{3C_A}\left(\frac{A_G^{\tau 2}}{A_q^{\tau 2}} + \frac{C_F}{C_A}\right)\right\}\,, \tag{84a}$$

$$\lambda_G^{\mathrm{eff},R}(z,Q_0^2) \;=\; \frac{16}{5\beta_0^2}\,\frac{\mathrm{a}_G^{\tau 4}}{a_s(Q_0^2)Q_0^2}\,, \tag{84b}$$

$$\lambda_{F2}^{\mathrm{eff},R}(z,Q_0^2) \;=\; \frac{64 C_F T_R f}{15\beta_0^2}\,\frac{1}{Q_0^2}\left[\mathrm{a}_q^{\tau 4}\left\{ \ln\left(\frac{Q_0^2}{z_q^2\,|\mathrm{a}_G^{\tau 4}|}\right) \;+\; \frac{4T_R f}{3C_A}\left(\frac{A_G^{\tau 2}}{A_q^{\tau 2}} + \frac{C_F}{C_A}\right)\right\} \;+\; \frac{\mathrm{a}_G^{\tau 4}}{C_F}\frac{A_G^{\tau 2}}{A_q^{\tau 2}}\right]\,. \tag{84c}$$

Because $\mathrm{a}_G^{\tau 4} < 0$, it is easy to see that $\lambda_{F2}^{\mathrm{eff},R}(z,Q_0^2) < \lambda_q^{\mathrm{eff},R}(z,Q_0^2)$. This indicates that the inequality $\lambda_{F2}^{\mathrm{eff}}(z,Q^2) > \lambda_q^{\mathrm{eff}}(z,Q^2)$ seems to be correct only at quite large Q^2 values (see also the previous section and discussions therein), where the twist-two terms give basic contributions.

Note also, that at Q_0^2 the slope $\lambda_q^{\mathrm{eff},R}(z,Q_0^2)$ rises at $x \to 0$, but the gluon slope $\lambda_G^{\mathrm{eff},R}(z,Q_0^2)$ is negative and x-independent. Thus, $\lambda_q^{\mathrm{eff},R}(z,Q^2) > \lambda_G^{\mathrm{eff},R}(z,Q_0^2)$ at low Q^2 that is in full agreement with the recent experimental data from HERA (see, for example, the review [15]). The twist-six terms do not change the above results essentially.

X. RESULTS OF THE FITS

With the help of the results obtained in the previous sections we have analyzed $F_2(x,Q^2)$ HERA data at small x from the H1 [1–6] and ZEUS [7–14] collaborations as separately, as well as together.

Without higher-twist corrections our solution of the DGLAP equations depends on five parameters, i.e. Q_0^2, x_0, $A_G^{\tau 2}$, $A_q^{\tau 2}$ and Λ (or, equally well, on $\alpha_{\mathrm{s}}(M_{\mathrm{Z}})$). The incorporation of twist-four and twist-six corrections leads to two and four additional parameters, respectively.

In order to keep the analysis as simple as possible we have fixed $\Lambda_{\overline{\rm MS}}$ to the values given in Eq. (85), which corresponds to $\alpha_s(M_Z) = 0.1166$, obtained recently by ZEUS [7]. The analyzed data region was restricted to $x < 0.01$ to stay in the kinematic region where our results are expected to be applicable. The χ^2 minimizations were done with MINUIT [60]. In the fits the errors are statistical and systematical added in quadrature. Finally, the number of active flavors was fixed to $f = 3$ and 4 for comparison.

A. Leading twist approximation

Tables II and III summarize the results of the fits to H1 and ZEUS data using twist-two formulas at LO (16) and NLO (25) approximations.

We can see in Tables II and III and in Fig. 1 that the qualities of the fits are very similar for the LO and NLO approximations. This suggest that perturbation theory works well in the small x regime. This is in accord with Refs. [61–63] (see also recent review [64]), where it was shown, that the argument of the strong coupling constant is effectively much larger as Q^2 in the small x domain.

Hovewer the similarity of the results found at LO and NLO fits does not agree with our previous analysis [44], where NLO corrections essentially improved the comparison between QCD and experiment. This disagreement relates mostly to the incorrect use of the same value of the QCD parameter Λ in [44] in both LO and NLO cases. By contrast, Λ should be different (see [65]). They are extracted from $\alpha_s(M_Z)$ by using b- and c-quarks thresholds following to [66]. The values of Λ obtained by this procedure and used hereafter in all the fits are:

$$\begin{aligned} \Lambda_{\rm LO}(f=5) &= 80.80 \text{ [MeV]}\,, & \Lambda_{\rm LO}(f=4) &= 111.8 \text{ [MeV]}\,, & \Lambda_{\rm LO}(f=3) &= 136.8 \text{ [MeV]}\,, \\ \Lambda_{\overline{\rm MS}}(f=5) &= 195.7 \text{ [MeV]}\,, & \Lambda_{\overline{\rm MS}}(f=4) &= 284.0 \text{ [MeV]}\,, & \Lambda_{\overline{\rm MS}}(f=3) &= 347.2 \text{ [MeV]}\,, \end{aligned} \qquad (85)$$

obtained from ZEUS result $\alpha_s(M_Z) = 0.1166$ (see [7]).

Table II contains the results of separate fits to H1 and ZEUS data with a low Q^2 cut, $Q^2_{\rm cut}$, that increases step by step. We observe that the agreement between theory and experiment improves when increasing the value of $Q^2_{\rm cut}$. For $Q^2 \geq 2.5$ GeV^2 the agreement is good (see Tables II and III).

Note that the separated fits of H1 and ZEUS data lead to purely comparable values of the parameters Q_0^2, x_0, $A_G^{\tau 2}$, $A_q^{\tau 2}$. Thus we may fit to the combined data set. The results of such combined fits can be found in the last rows of Table II and Table III.

Looking carefully on that Tables, we arrive to the following conclusions:

- In the leading twist approximation the preferred number of flavors f is four.
- The value of the quark distribution does not depend on the specific $Q^2_{\rm cut}$ values within the limits of experimental errors. The magnitude of the gluon density and Q_0^2 decrease slowly with decreasing $Q^2_{\rm cut}$.
- A strong reduction of the magnitude of the gluon density is observed when NLO corrections are included.

  The suppression of the gluon density rise with Q^2 at NLO in comparison with the LO prediction is well-known effect [59, 67] but in addition we also observe a strong reduction of the gluon magnitude at Q_0^2.

  At least partialy, this effect can be explained based on the GRV-like point of view [31–34], where at low Q^2 values there are only valence quarks and all other types of partons are generated in the Q^2-evolution. Thus, the slowe rise with Q^2 when NLO corrections are included directly implies a reduction of the magnitude at a given Q_0^2.

  It should be mentioned that a similar relative reduction of gluon normalization is obtained in the analyses [35, 68], when the $\ln(1/x)$ resummation was included. Thus, the correct incorporation of NLO terms has a similar tendency.
- The fitted Q_0^2 values are essentially higher at NLO: $Q_0^2 \sim 0.5 \div 0.6$ GeV^2, in comparison with LO fits, where $Q_0^2 \sim 0.3 \div 0.4$ GeV^2, and comparable to those obtained earlier in [44].

  Partialy, the effect can be explained by different Λ values at LO and NLO approximations. Note, however, that the ratio $\Lambda^2_{\overline{\rm MS}}/\Lambda^2_{\rm LO} \sim 6.4$ and, thus, the Q^2 dependence of F_2 data itself should be important in the definition of Q_0^2.

Considering Tables II and III and Fig. 1 we find good agreement with data only at $Q^2 \geq 2.5$ GeV^2. The situation is little bit worse than it was before in [44], mainly due to the strong improvement of experimental data. To expand the range of applicability of our analysis to $Q^2 < 2.5$ GeV^2 we add to our fits HT corrections presented in the previous sections.

Let's consider both types of estimations of the HT corrections separately.

B. BFKL-motivated estimations for twist-four operators

Tables IV–VI and Fig. 4 contain the results of the fits to H1 and ZEUS data using Eqs. (16), (50) and (53) at LO and (25), (50) and (53) at NLO.

The results demonstrate good agreement between the theoretical predictions having BFKL-like twist-four term and experimental data of the H1 [3, 4] and ZEUS [11, 12] collaborations.

The fits of H1 [3, 4] and ZEUS [11, 12] data demonstrate a strong improvement of the agreement between theory and experiment (see Fig. 4), essentially at LO and in the case $f = 4$.

The values of parameters in the twist-two terms do not change drastically. Q_0^2 rises 100 MeV and 150 MeV at LO and NLO, respectively. The gluon density in the twist-two term rises essentially and the quark distribution decreases slowly. The changes are compensated by a negative gluon and a positive quark twist-four magnitudes, respectively.

We found also a tiny dependence on the real value of the parameter 'b', that supports our hypothese (see Section III) about the irrelevance of the exact form for the nonsingular (at $n \to 1$) terms in the twist-four anomalous dimensions.

An interesting fact is that the value of the sum $A_G^{\tau 4} + 4/9 A_q^{\tau 4}$ is very close to zero. Hence, HERA data do not seem to support a strong increase of the twist-four terms at small x, contrary to the expectation from various BFKL-motivated estimations [46–49]. However a small value for the the twist-four terms has also been found in a model-dependent analysis [69].

C. Renormalon model predictions for higher twist operators

Tables IV–VII and Figs. 4 and 5 contain the results of the fits to H1 and ZEUS data using Eqs. (16) and (66) at LO and (25) and (66) at NLO. The results demonstrate excellent agreement between theoretical predictions and experimental data. The χ^2 decreases very strongly.

Consider separately the fits of data for $Q^2 \geq 1.5$ GeV^2 and $Q^2 \geq 0.5$ GeV^2, presented in Tables IV–VI (and on Fig. 4) and Table VII (and on Fig. 5), respectively.

Looking carefully Tables IV–VI and Fig. 4, we arrive to the following conclusions:

- For the data usage of $f = 4$ is strongly preffered.

- The values of parameters in the twist-two terms do not change essentially.

  We see, however, for H1 data in Table IV and for combibed data in Table VI some rise of gluon terms when higher twist terms are incorporated. The rise exists for both the LO and NLO approximations and it is compensated by negative gluon twist-four magnitude. The twist-six gluon magnitude has different signs (it is negative and positive at LO and NLO approximations, respectively) but the combination of the higher twist terms gives negative contribution for the gluon case.

  Note that the phenomenon is similar to one observed for BFKL-motivated twist-four corrections (see previous subsection) and can be considered as quite general property of the HT corrections.

- For the ZEUS data in Table V the influence of the higher twist terms is not so important.

- In contrary to the gluon case, the higher twist corrections for the quark density are mostly positive that leads to different small-x asymptotics of gluon and quark distributions at low Q^2 values, observed recently at HERA experiments [70] (see a detailed discussions in the subsection F).

- The fitted value of Q_0^2 tends to be little higher (at LO $Q_0^2 \sim 0.5$ GeV^2 and at NLO $Q_0^2 \sim 0.7 \div 0.8$ GeV^2) when the twist-four corrections have been added. It is in agreement with the results when BFKL-motivated twist-four corrections have been considered (see the previous sunbsection). The incorporation of twist-six terms returns the Q_0^2 values to the ones, obtained in the twist-two approximation.

Looking carefully Table VII and Fig. 5, we see full support of above results: the agreement with experimental data improves drastically, essentially for 0.5 GeV$^2 \leq Q2 \leq 2.5$ GeV^2. We should note, however, about following excepting features:

- Usage of $f = 3$ is preffered, that is natural choice at low Q^2 values.

- The twist-six corrections are important to stabilize the HT contributions and, thus, the results of Table VII are comparable with ones in Tables IV–VI only when the twist-six corrections taken into account.

D. Leading and higher twist approximations for the derivative $\partial F_2/\partial \ln Q^2$

The results for the derivative $\partial F_2/\partial \ln Q^2$ are shown on Fig. 3 and Fig. 8 together with H1 experimental data [2].

Fig. 3 contains only the leading twist theoretical predictions. As in the case of F_2 data we have very good agreement between our formulae and experimental data at $Q^2 \geq 3$ GeV^2.

When we added the HT corrections, the theoretical results begin to be in agreement with experiment also at $Q^2 < 3$ GeV^2 (see Fig. 8), especially when we used the results of F_2 data fits at $Q^2 \geq 0.5$ GeV^2. The corresponding results for $\partial F_2/\partial \ln Q^2$ are shown as the dashed curve for the NLO and as the dash-dotted curve for the LO fits.

Both curves are hardly distinguished from each other. It meens, that in this kinematical region of small x the order of perturbation theory inside the leading twist does not matter. The importance has the number of twists taking into account.

Note that the HT corrections to F_2 and $\partial F_2/\partial \ln Q^2$ structure functions are opposite in sign that demonstrates the importance, respectively, the quark density and gluon one for the functions (see also the following subsection and discussions therein). The fact is in full agreement with results of Section VI.

Thus, our quite simple formulas obtained in the generalized DAS approach are very convenient also to the study the derivative $\partial F_2/\partial \ln Q^2$, which is very important to extract gluon density and the longitudinal F_L or the ration $R = \sigma_L/\sigma_T$ (see [71, 72] and [73–75], respectively).

E. Effective slope $\lambda^{\rm eff}_{F_2}(x, Q^2)$

The results for the slope $\lambda^{\rm eff}_{F_2}(x, Q^2)$ are shown on Figs. 2, 6 and 7 together with H1 and ZEUS experimental data [1, 8, 76, 77].

At Figs. 2 and 6 we see very good agreement between theory and experiment as with and without consideration of the HT corrections. Note that the asymptotic approximation does not work so well because at large Q^2 values, i.e. at its range of applicability, there are experimental data only at quite large x values: $x > 10^{-3}$.

Since the logarithmic x derivative is compatible with independence of Q^2, H1 and ZEUS have both fitted their data on the proton structure function to the form $F_2 = c(Q^2)\, x^{-\lambda(Q^2)}$. Fig. 7 shows recent H1 and ZEUS fits [1, 8, 9, 76] for $\lambda(Q^2)$. Some of them are preliminary only and extracted from Fig. 14 of the recent review [77].

The experimental data shows a rise of the slope $\lambda^{\rm eff}_{F_2}(x, Q^2)$ from the value ~ 0.1 at $Q^2 \leq 1$ GeV^2 (so-called *"soft pomeron range"*) to the value $\sim 0.3 \div 0.4$ at $Q^2 \geq 100$ GeV^2 (so-called *"hard pomeron range"*) and $c \sim 0.18$ is consistent with being constant.

In our opinion, the strong Q^2 dependence of the slope $\lambda^{\rm eff}_{F_2}(x, Q^2)$ was observed firstly in Ref. [78], where fits of experimental data have been performed for the Regge-like PD form. At high Q^2 side, the slope value is close to LO BFKL prediction ($\lambda^{\rm eff}_{F_2}(x, Q^2) \sim 0.3 \div 0.4$), at smaller Q^2 values $\lambda^{\rm eff}_{F_2}(x, Q^2) \sim 0.2$, that is close to model with Pomeron interactions [79] and to NLO BFKL predictions [63] based on non-$\overline{\rm MS}$-like renormalization schemes and BLM resummation of large values of NLO corrections calculated recently in [80, 81] (see also [82, 83]). At low Q^2 the slope value coincides with Donnachie-Landshoff model, where $\lambda^{\rm eff}_{F_2}(x, Q^2) \sim 0.1$.

In a sence, the shape of the slope $\lambda^{\rm eff}_{F_2}(x, Q^2)$ is in contrast with Regge asymptotics, where the corresponding slopes should be Q^2-independent. Note, however, that this Q^2-dependence can be described in phenomenological Regge-like models [84–87]. There are also attempts (see [39, 40]) to recove the slope shape in the Regge-like form of parton distributions considering the small x asymptotics of DGLAP equation. A quite natural explanation of the rise is given in the generalized DAS approximation as it was shown in [44].

Quite recently the H1 96/97 data [1] (black circles on Fig. 7) has been analysed in [53], where good agreement has been found between data and theoretical predictions based on generalized DAS approach. For example, the rise can be described as $\ln \ln Q^2$, i.e. in pure perturbative QCD. Incorparation of HT corrections gives a possibility to extend the agreement to new preliminaty H1 and ZEUS data for quite low Q^2 values (see dashed curve and the preliminary data near $Q^2 \sim 1$ GeV^2 on Fig. 7).

F. Parton distributions

The results for the quark and gluon densities are shown on Fig. 9 together with the NLO QCD predictions of A02NLO [88], represented by dots.

As it was noted already in Sections VII and IX, there is very strong difference between the twist-two and total parton distributions. In the case of the twist-two parton densities $f^{\tau 2}_a(x, Q^2)$ the higher-twist corrections contribute to the Wilson coeffcient functions, i.e. (in $\overline{\rm MS}$-like factorization scheme used here) to the relation between the parton

distributions $f_a^{\tau 2}(x,Q^2)$ and F_2. Then, the higher-twist terms give additional power-like corrections to the relation and, thus, change it.

Contrary to this, in the case of the total parton densities $f_a(x,Q^2)$ the coefficient functions are pure twist-two ones, i.e. the relation between the parton distributions $f_a(x,Q^2)$ and the structure function F_2 taken in the standard $\overline{\rm MS}$-like way. Thus, in the case the higher-twist corrections are responsable for the difference between the twist-two parton distributions $f_a^{\tau 2}(x,Q^2)$ and the full ones $f_a(x,Q^2)$.

For the quark density the difference between twist-two distributions and total densities are not very strong. In Fig. 9 one can see good agreement between quark distributions obtained in the different approximations. For the renormalon higher-twist corrections, our results very close to obtained by ZEUS Collaboration in [70].

At high Q^2 values there is also good agreement between gluon distributions obtained in the different approximations. For small Q^2 values, in the renormalon model our total gluon density is strictly less then twist-two one: for example, at $Q^2 = 2$ GeV^2 the ration $f_G(x,Q^2)/f_G^{\tau 2}(x,Q^2) < 1/3$. Neverseless, there is a disagreement here between our results and the recent one from ZEUS Collaboration (see [70]) in the range of small Q^2 values: our total gluon density is higher essentially the ZEUS one. A similar disagreement exist between our total gluon distribution and Alekhin one [88] (see Fig. 9). Thus, the deviation between our twist-two and total gluon distributions is strong but less to have agreement with experimental data.

In our opinion, most part of the difference comes from neglection of valent quark part $f_V(x,Q^2)$ in our article. The neglection is a quite standart tool at small x range (and quite large Q^2 values, where parton model is applicable), because $f_V(x,Q^2) \sim x^{\lambda_V}$ with $\lambda_V \sim 0.3 \div 0.5$.

At low Q^2 values, however, the ignoring the valent and nonsinglet quark distributions cannot be the correct approximation, because here the singlet parton distributions (at least the gluon density) start to fall when $x \to 0$. Moreover, at higher orders of perturbation theory strong double-logarithmic terms contributte to the valent and nonsinglet quark distributions. The contributions can be evaluated in the framework of BFKL-like approach and they can lead to essential decreasing of the λ_V value at low Q^2 values (see [89, 90] and discussion therein).

Thus, in our model gluon density at small Q^2 values includes effectively a contribution of the valent quark distributions and, thus, is large essentially to compare with ZEUS and Alekhin predictions from [70] and [88], respectively.

Note that the absence of the valent quarks can be partially responsable for some disagreement between theory and experiment for the derivation $\partial F_2/\partial \ln Q^2$ (see Figs. 3 and 8 and discussions in subsection D), which is depended strongly on gluon density.

We plan to return to the study the problem and to incorporate the valent quark densities in our future investigations.

XI. CONCLUSIONS

In generalized DAS approximation we have incorporated HT corrections for semi-analytical solution of DGLAP equation obtained earlier in [44] at LO and NLO levels in the leading twist approximation for the flat initial condition.

The HT corrections have been added in two models, the so-called BFKL-like one and the renormalon one. In both models the HT terms lead to improvement of the agreement with new precise experimental data of H1 and ZEUS Collaborations. The elements of the renormalon model, however, are essentially better defined and the model describes experimental data much better, especially at very low Q^2 values ($Q^2 \geq 0.5$ GeV^2).

After verification of all uncalculable parameters in our formulae from the fits of F_2 data we apply our approach to compare with H1 data for the derivative $\partial F_2/\partial \ln Q^2$, with H1 and ZEUS data for the effective slope $\lambda^{\rm eff}_{F_2}(x,Q^2)$ data and with experimental predictions for the parton distributions.

We have found rather good agreement with data for the effective slope $\lambda^{\rm eff}_{F_2}(x,Q^2)$ and for the derivative $\partial F_2/\partial \ln Q^2$ and also with experimental predictions for the quark distribution, but have some disagreement with other results for gluon densities at low Q^2 values (see the subsection F in the previous section and discussions therein), that needs an additional investigations.

As next steps we plan to add at low Q^2 values to our analysis some phenomenological models of coupling constant. We hope to apply the Shirkov-Solovtsov analitization [91, 92] and a "freezing" procedure (see, for example, Ref. [93] and discussions therein).

Moreover we plan also to add to our initial conditions (15) corrections $\sim \ln(1/x)$ and $\sim \ln^2(1/x)$ obeying to Froassart restriction by analogy with consideration of these corrections in the Regge-like small-x asymptotics of parton distributions done earlier in [94, 95].

Addition of HT terms should be important also for high-energy cosmic rays, where they can lead to quite important shadowing corrections for cross-sections of neutrino-proton scattering studyed in DAS approach in [96]. The subject will be considered in forthcoming article.

We are considering also to extend the application of the higher twist corrections for the longitudinal srtucture function F_L. The consideration of F_L should be very important essentially at low Q^2 values, where F_L should go to

zero when $Q^2 \to 0$ [97–100] at low x values based on k_t-factorization procedure [101–104]).

In the QCD improved parton model the LO results for $F_L \sim \alpha_s(Q^2)$ and, thus, do not lead to zero values to the longitudinal srtucture function. Moreover, the NLO corrections to F_L are large and negative at low x values (see [105–112] and, thus, give large negative contributions at low Q^2 range [62, 113–115]. Thus, they can lead to the negative values for F_L [29, 62] of perturbation theory and needs a resummation of large corrections at low Q^2 values. Based on Grunberg approach [116, 117], the resummation leads to recovering well-know Callan-Gross relation $F_L = 0$ at asymptotics $x \to 0$ (see [62]).

Thus, there are a quite conserval results for F_L at low x and Q^2 values. The incorporation of the higher-twist corrections, which can be very important namely in the case of the longitudinal srtucture function (see recent study [69] and discussions therein), should give an additional important information about F_L structure at low x and Q^2 values. Moreover, the measurement of F_L should become possible in nearest future (see discussions in Section VII of [15]) with the proposed updates to the HERA machine, which will yield very large integrated luminosity. Note that some precise preliminary results for F_L can be found already in the recent review [77] and we plan to study of them in nearest future.

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XII. APPENDIX A

The twist-four and twist-six contributions in the framework of IR-renormalon model have been calculated in [118] (for nonsinglet case) and in [50] (for singlet one). As we already noted in Section II, we neglect the nonsinglet component in our analysis. The higher-twist corrections to singlet case contain sum of nonsinglet and singlet higher-twist results. However, we have interest only to $n \to 1$ asymptotics of the corrections, where nonsinglet part of higher-twist corrections are neglected because exact Bjorken sum rule.

The singlet part of higher-twist corrections may be presented in the following form

$$M_a^R(n,Q^2) \;=\; M_a(n,Q^2)\left[1+\frac{\mathrm{a}_a^{\tau 4}}{Q^2}\mu_a^{\tau 4}\left(n,\ln\left(\frac{|\mathrm{a}_a^{\tau 4}|}{Q^2}\right)\right)+\frac{\mathrm{a}_a^{\tau 6}}{Q^4}\mu_a^{\tau 6}\left(n,\ln\left(\frac{\sqrt{|\mathrm{a}_a^{\tau 6}|}}{Q^2}\right)\right)\right]\,. \tag{A1}$$

The quark contributions $\mu_q^{\tau 4}(n, \ln(\mathrm{A}/Q^2))$ and $\mu_q^{\tau 6}(n, \ln(\mathrm{A}/Q^2))$ [50] may be transformed to n-space

$$\mu_a^{\tau m}\left(n,\ln\left(\frac{\mathrm{A}}{Q^2}\right)\right) \;=\; \frac{8C_F T_R f}{\beta_0^2}\left[B_a^{\tau m}(n)+b_a^{\tau m}(n)\;\ln\left(\frac{\mathrm{A}}{Q^2}\right)\right]\,, \tag{A2}$$

where

$$B_q^{\tau 4}(n) = \frac{16}{15}\frac{1}{(n-1)^2} + \frac{22}{225}\frac{1}{n-1} + \frac{11}{3}\frac{1}{n+1} - \frac{25}{9}\frac{1}{n+2} - \frac{74}{75}\frac{1}{n+4}$$
$$- \frac{12}{(n+1)^2} + \frac{6}{(n+2)^2} + \frac{4}{(n+1)^3} + \frac{12}{(n+2)^3} \ , \tag{A3}$$
$$b_q^{\tau 4}(n) = -\frac{8}{15}\frac{1}{n-1} + \frac{9}{n+1} - \frac{23}{3}\frac{1}{n+2} - \frac{4}{5}\frac{1}{n+4} - \frac{2}{(n+1)^2} - \frac{6}{(n+2)^2} \ , \tag{A4}$$

$$B_q^{\tau 6}(n) = -\frac{128}{105}\frac{1}{(n-1)^2} - \frac{572}{11025}\frac{1}{n-1} + \frac{52}{75}\frac{1}{n+1} + \frac{32}{9}\frac{1}{n+2} + \frac{16}{3}\frac{1}{n+3}$$
$$- \frac{724}{75}\frac{1}{n+4} + \frac{452}{3675}\frac{1}{n+6} + \frac{16}{5}\frac{1}{(n+1)^2} - \frac{16}{(n+3)^2} \ . \tag{A5}$$
$$b_q^{\tau 6}(n) = \frac{64}{105}\frac{1}{n-1} - \frac{8}{5}\frac{1}{n+1} - \frac{8}{3}\frac{1}{n+2} + \frac{8}{n+3} - \frac{24}{5}\frac{1}{n+4} + \frac{16}{35}\frac{1}{n+6} \ , \tag{A6}$$

The gluon contributions $\mu_G^{\tau 4}(n, \ln(\mathrm{A}/Q^2))$ and $\mu_G^{\tau 6}(n, \ln(\mathrm{A}/Q^2))$ may be estimated [50] as

$$\mu_G^{\tau m}(n, \ln(\mathrm{A}/Q^2)) \ = \ \mu_q^{\tau m}(n, \ln(\mathrm{A}/Q^2)) \ / \ \gamma_{Gq}^{(0)}(n) \ , \tag{A7}$$

where [119]

$$\gamma_{Gq}^{(0)}(n) \ = \ -4C_F \, \frac{2+n+n^2}{(n-1)n(n+1)}$$

is the leading contribution to the gluon-quark anomalous dimension.

We have the interest to the asymptotics $n \to 1$, where the above values may be represented as

$$B_q^{\tau 4}(n) = -\frac{4}{15}\left(\frac{1}{\delta^2} + \frac{11}{120}\frac{1}{\delta} - \frac{2291}{3600}\right) + \mathcal{O}(\delta) \ , \quad b_q^{\tau 4}(n) = \frac{2}{15}\left(\frac{1}{\delta} - \frac{139}{120}\right) + \mathcal{O}(\delta) \ ,$$
$$B_q^{\tau 6}(n) = \frac{32}{105}\left(\frac{1}{\delta^2} + \frac{143}{3360}\frac{1}{\delta} - \frac{870637}{1411200}\right) + \mathcal{O}(\delta) \ , \quad b_q^{\tau 6}(n) = -\frac{16}{105}\left(\frac{1}{\delta} - \frac{3217}{3360}\right) + \mathcal{O}(\delta) \ ; \tag{A8}$$

$$B_G^{\tau 4}(n) = -\frac{2}{15C_F}\left(\frac{1}{\delta} + \frac{101}{120}\right) + \mathcal{O}(\delta) \ , \quad b_G^{\tau 4}(n) = \frac{1}{15C_F} + \mathcal{O}(\delta) \ ,$$
$$B_G^{\tau 6}(n) = \frac{16}{105C_F}\left(\frac{1}{\delta} + \frac{2663}{3360}\right) + \mathcal{O}(\delta) \ , \quad b_G^{\tau 6}(n) = -\frac{8}{105C_F} + \mathcal{O}(\delta) \ , \tag{A9}$$

with $\delta = n - 1$.

XIII. APPENDIX B

We present here the detailed analysis [137] of the method of replacing the convolution of two functions by a simple product at small x. We restrict ourselves to the accuracy $\mathcal{O}(z)$. Some earlier presentations can be found in [54, 55] (here the accuracy $\mathcal{O}(z^2)$ has been considered) and in [44].

Let us to consider the set of PD with different forms:

(I) Regge-like form $f_R(z) = z^{-\delta}\widetilde{f}(z)$,

(II) Logarithmic-like form $f_L(z) = z^{-\delta}\ln(1/z)\widetilde{f}(z)$,

(III) Bessel-like form $f_I(z) = z^{-\delta}\, \hat{d}\ln(1/z)^{k/2}\widetilde{I}_k\left(2\sqrt{\hat{d}\ln(1/z)}\right)\widetilde{f}(z)$ with definition (22) of the $\widetilde{I}_k$ function,

where $\widetilde{f}(z)$ and its derivative $\widetilde{f}'(z) \equiv d\widetilde{f}(z)/dz$ are smooth at $z = 0$ and both are equal to zero at $z = 1$:

$$\widetilde{f}(1) \ = \ \widetilde{f}'(1) \ = \ 0 \ .$$

(1) At the beginning we consider the basic integral with integer integer nonnegative n values:

$$J^{(1)}_{\delta,i}(n,z) = z^n \otimes f_i(z) \equiv \int_z^1 \frac{\mathrm{d}y}{y}\, y^n f_i\left(\frac{z}{y}\right) \quad , \quad i = R, L, I \; .$$

(a) Regge-like case. Expanding $\widetilde{f}(z)$ near $\widetilde{f}(0)$, we have

$$\begin{aligned} J^{(1)}_{\delta,R}(n,z) &= z^{-\delta} \int_z^1 \mathrm{d}y\; y^{n+\delta-1} \left[\widetilde{f}(0) + \frac{z}{y}\widetilde{f}^{(1)}(0) + \ldots + \frac{1}{k!}\left(\frac{z}{y}\right)^k \widetilde{f}^{(k)}(0) + \ldots \right] \\ &= z^{-\delta}\left[\frac{1}{n+\delta}\; \widetilde{f}(0) + \mathcal{O}(z)\right] \\ &- z^n \left[\frac{1}{n+\delta}\; \widetilde{f}(0) + \frac{1}{n+\delta-1}\; \widetilde{f}^{(1)}(0) + \ldots + \frac{1}{k!}\; \frac{1}{n+\delta-k}\; \widetilde{f}^{(k)}(0) + \ldots\right] \; . \end{aligned} \tag{B1}$$

Using the power-like large x asymptotics

$$f(z) \;\sim\; (1-z)^\nu \text{ when } z \to 1 \; , \tag{B2}$$

the second term on the r.h.s. of Eq. (B1) can be summed:

$$J^{(1)}_{\delta,R}(n,z) = z^{-\delta}\left[\frac{1}{n+\delta}\; \widetilde{f}(0) + \mathcal{O}(z)\right] \; + \; z^n\; \frac{\Gamma(-(n+\delta))\Gamma(1+\nu)}{\Gamma(1+\nu-n-\delta)}\; \widetilde{f}(0) \; . \tag{B3}$$

Consider particular cases $n \geq 1$ and $n = 0$ separately:
(a1) If $n \geq 1$ then the second term in the r.h.s. of (B3) is negligible and we have

$$J^{(1)}_{\delta,R}(n,z) = z^{-\delta}\frac{1}{n+\delta}\; \widetilde{f}(0) + \mathcal{O}(z^{1-\delta}) \;=\; \frac{1}{n+\delta}\; \widetilde{f}_R(z) + \mathcal{O}(z^{1-\delta}) \; . \tag{B4}$$

(a2) If $n = 0$, the r.h.s. of (B3) can be rewritten as follows

$$\begin{aligned} J^{(1)}_{\delta,R}(0,z) &= z^{-\delta}\left[\frac{1}{\delta}\widetilde{f}(0) + \mathcal{O}(z)\right] \; + \; \frac{\Gamma(-\delta)\Gamma(1+\nu)}{\Gamma(1+\nu-\delta)}\; \widetilde{f}(0) \\ &= \delta_R^{-1}(z) f_R(z) + \mathcal{O}(z^{1-\delta}) \; , \end{aligned} \tag{B5}$$

where

$$\frac{1}{\delta_R(z)} \;=\; \frac{1}{\delta}\left[1 - \frac{\Gamma(1-\delta)\Gamma(1+\nu)}{\Gamma(1+\nu-\delta)} z^\delta\right], \tag{B6}$$

i.e. there is the correlation between small x and large x asymptotics of parton distributions (see [120–122]. Note that the value $\delta_R^{-1}(z)$ is finite at the limit $\delta \to 0$:

$$\lim_{\delta\to 0} \frac{1}{\delta_R(z)} \;=\; \ln\left(\frac{1}{z}\right) - \left[\Psi(1+\nu) - \Psi(1)\right] \;\equiv\; \ln\left(\frac{1}{z}\right) - p(\nu) \; , \tag{B7}$$

where the Riemannian Ψ-function is the logarithmic derivation of the Γ-function.

Remember that the large x asymptotics are different in quark and gluon cases, the values $\nu_q \approx 3$ and $\nu_G \approx 4$ are coming from quark counting rules, what lead to $p(\nu_q) \approx 11/6$ and $p(\nu_G) \approx 25/12$.
(b) Logarithmic-like case. Using the simple relation

$$z^{-\delta} \ln(1/z) = d(z^{-\delta})/d\delta$$

we immediately obtain:
(b1) $n \geq 1$ case:

$$\begin{aligned} J^{(1)}_{\delta,L}(n,z) &= z^{-\delta} \ln(1/z) \left[\frac{1}{n+\delta}\left(1 - \frac{1}{(n+\delta)\ln(1/z)}\right) \widetilde{f}(0) + \mathcal{O}(z)\right] \\ &= \frac{1}{n+\delta}\left(1 - \frac{1}{(n+\delta)\ln(1/z)}\right) f_L(z) + \mathcal{O}(z^{1-\delta}) \\ &= \frac{1}{n+\delta}\; f_L(z) + \mathcal{O}\left(\frac{1}{\ln(1/z)}\right) \; . \end{aligned} \tag{B8}$$

(b2) $n = 0$ case:

$$J^{(1)}_{\delta,L}(0,z) = z^{-\delta}\ln(1/z)\left[\frac{1}{\delta}\left(1-\frac{1}{\delta\ln(1/z)}\right)\widetilde{f}(z)+\mathcal{O}(z)\right] + \frac{\Gamma(-\delta)\Gamma(1+\nu)}{\Gamma(1+\nu-\delta)}\,\widetilde{f}(0)\left[\Psi(1+\nu-\delta)-\Psi(-\delta)\right] = \delta_L^{-1}(z)\; f_L(z)+\mathcal{O}(z^{1-\delta})\ , \tag{B9}$$

where

$$\frac{1}{\delta_L(z)} \equiv \frac{z^{\delta}}{\ln(1/z)}\frac{d}{d\delta}\left(\frac{z^{-\delta}}{\delta_R(z)}\right) = \frac{1}{\delta_R(z)}+\frac{1}{\ln(1/z)}\frac{d}{d\delta}\left(\frac{1}{\delta_R(z)}\right) = \frac{1}{\delta}\left[1-\frac{1}{\ln(1/z)}\left(\frac{1}{\delta_R(z)}+\frac{\Gamma(1-\delta)\Gamma(1+\nu)}{\Gamma(1+\nu-\delta)}z^{\delta}\left[\Psi(1+\nu-\delta)-\Psi(1-\delta)\right]\right)\right]\ . \tag{B10}$$

The value $\delta_L^{-1}(z)$ is also finite at the limit $\delta\to 0$:

$$\lim_{\delta\to 0}\frac{1}{\delta_L(z)} = \frac{1}{2}\ln\left(\frac{1}{z}\right)-\frac{1}{2\ln(1/z)}\left(\left[\Psi(1+\nu)-\Psi(1)\right]^2-\left[\Psi'(1+\nu)-\Psi'(1)\right]\right) = \frac{1}{2}\ln\left(\frac{1}{z}\right)-\frac{1}{2\ln(1/z)}\left(p(\nu)^2-p'(\nu)\right)\ , \tag{B11}$$

where the Ψ'-function is the derivation of the Ψ-function, $p'(\nu_q)\approx -49/36$ and $p'(\nu_G)\approx -205/144$ are coming from quark counting rules.
(c) Bessel-like case. Representing Bessel function in the form

$$z^{-\delta}\,\hat{d}\ln(1/z)^{k/2}\widetilde{I}_k\left(2\sqrt{\hat{d}\ln(1/z)}\right) = \sum_{n=0}^{\infty}\frac{1}{n!\Gamma(n+k+1)}\left(\hat{d}\frac{d}{d\delta}\right)^{n+k} z^{-\delta} = \hat{d}\left(\frac{d}{d\delta}\right)^{k/2}\widetilde{I}_k\left(2\sqrt{\hat{d}\left(\frac{d}{d\delta}\right)}\right)z^{-\delta} \tag{B12}$$

and repeating the above analysis, we have
(c1) in the $n\geq 1$ case:

$$J^{(1)}_{\delta,I}(n,z) = \frac{1}{n+\delta}\;f_I(z)+\mathcal{O}\left(\sqrt{\frac{\hat{d}}{\ln(1/z)}}\right)\ , \tag{B13}$$

(c2) in the $n=0$ case:

$$J^{(1)}_{\delta,I}(0,z) = \frac{1}{\delta_I(z)}\;f_I(z)+\mathcal{O}(z^{1-\delta})\ , \tag{B14}$$

where

$$\frac{1}{\delta_I(z)} = \frac{z^{\delta}\,\hat{d}\left(\frac{d}{d\delta}\right)^{k/2}}{\hat{d}\ln(1/z)^{k/2}\widetilde{I}_k\left(2\sqrt{\hat{d}\ln(1/z)}\right)}\widetilde{I}_k\left(2\sqrt{\hat{d}\left(\frac{d}{d\delta}\right)}\right)\frac{z^{-\delta}}{\delta_R(z)}\ . \tag{B15}$$

The value $\delta_I^{-1}(z)$ is also finite at the limit $\delta\to 0$:

$$\lim_{\delta\to 0}\frac{1}{\delta_I(z)} = \sqrt{\frac{\ln(1/z)}{\hat{d}}}\frac{\widetilde{I}_{k+1}\left(2\sqrt{\hat{d}\ln(1/z)}\right)}{\widetilde{I}_k\left(2\sqrt{\hat{d}\ln(1/z)}\right)} \approx \sqrt{\frac{\ln(1/z)}{\hat{d}}}-\frac{2k+1}{4\hat{d}}+\mathcal{O}\left(\sqrt{\frac{\hat{d}}{\ln(1/z)}}\right)\ , \tag{B16}$$

where the r.h.s. of (B16) is obtained from the expansion of the modified Bessel functions at $z\to 0$.

Note that we can represented the Eqs. (B5), (B9) and (B14) formally as follows

$$\delta^{-1}\, f_B(z) \;=\; \frac{1}{\delta_B(z)}\, f_B(z) \quad (B=R,L,I), \tag{B17}$$

which has been used in Sections III and V.

(2) Since the HT coefficient functions $B_q^{\tau 4,6}(n)$ contain the terms $\sim 1/(n-1)^2$ (see Eqs. (A3) and (A5)), we should consider also the second basic integral with integer nonnegative n values:

$$J^{(2)}_{\delta,i}(n,z) = z^n \ln(1/z) \otimes f_i(z) \equiv \int_z^1 \frac{\mathrm{d}y}{y}\, y^n \ln(1/y) f_i\left(\frac{z}{y}\right) \;, \quad i=R,L,I\;.$$

It is easy to demonstrate that

$$J^{(2)}_{\delta,i}(n,z) = \frac{d}{da}\, J^{(1)}_{\delta,i}(n-a,z)|_{a=0},$$

that symplyfies essentially the consideration of $J^{(2)}_{\delta,i}(n,z)$.

(a) Regge-like case. Repeating the analysis of the subsection (1a), we obtain easy that

$$\begin{aligned} J^{(2)}_{\delta,R}(n,z) \;&=\; z^{-\delta}\left[\frac{1}{(n+\delta)^2}\; \widetilde{f}(0) + \mathcal{O}(z)\right] \\ &+\; z^n\, \frac{\Gamma(-(n+\delta))\Gamma(1+\nu)}{\Gamma(1+\nu-n-\delta)}\; \left[\ln\frac{1}{z} + \Psi(-(n+\delta)) - \Psi(1+\nu-n-\delta)\right] \widetilde{f}(0)\;. \end{aligned} \tag{B18}$$

Consider particular cases $n \geq 1$ and $n=0$ separately:
(a1) If $n \geq 1$ then the second term in the r.h.s. of (B3) is negligible and we have

$$J^{(2)}_{\delta,R}(n,z) = z^{-\delta}\frac{1}{(n+\delta)^2}\; \widetilde{f}(0) + \mathcal{O}(z^{1-\delta}) \;=\; \frac{1}{(n+\delta)^2}\; \widetilde{f}_R(z) + \mathcal{O}(z^{1-\delta})\;. \tag{B19}$$

(a2) If $n=0$, the r.h.s. of (B18) can be rewritten as follows

$$\begin{aligned} J^{(2)}_{\delta,R}(0,z) &= z^{-\delta}\left[\frac{1}{\delta^2}\widetilde{f}(0) + \mathcal{O}(z)\right] \;+\; \frac{\Gamma(-\delta)\Gamma(1+\nu)}{\Gamma(1+\nu-\delta)}\left[\ln\frac{1}{z} + \Psi(-\delta) - \Psi(1+\nu-\delta)\right]\; \widetilde{f}(0) \\ &= \delta_R^{-2}(z) f_R(z) + \mathcal{O}(z^{1-\delta})\;, \end{aligned} \tag{B20}$$

where

$$\frac{1}{\delta^2_R(z)} \;=\; -\frac{d}{d\delta}\,\frac{1}{\delta}\left[1 - \frac{\Gamma(1-\delta)\Gamma(1+\nu)}{\Gamma(1+\nu-\delta)}z^{\delta}\right] \;\equiv\; -\frac{d}{d\delta}\,\frac{1}{\delta_R(z)}. \tag{B21}$$

Note that the value $\delta_R^{-2}(z)$ is finite at the limit $\delta \to 0$:

$$\lim_{\delta\to 0}\frac{1}{\delta^2_R(z)} \;=\; \frac{1}{2}\left[\left(\lim_{\delta\to 0}\frac{1}{\delta_R(z)}\right)^2 - p'(\nu)\right]\;, \tag{B22}$$

where the value of $\lim_{\delta\to 0}(1/\delta_R(z))$ is given in (B7).
(b) Logarithmic-like case. Following to the subsection (1b), we obtain:
(b1) $n \geq 1$ case:

$$J^{(2)}_{\delta,L}(n,z) = \frac{1}{(n+\delta)^2}\; f_L(z) + \mathcal{O}\left(\frac{1}{\ln(1/z)}\right)\;. \tag{B23}$$

(b2) $n=0$ case:

$$J^{(1)}_{\delta,L}(0,z) \;=\; \delta_L^{-2}(z)\; f_L(z) + \mathcal{O}(z^{1-\delta})\;, \tag{B24}$$

where

$$\frac{1}{\delta_L^2(z)} \;=\; -\frac{d}{d\delta}\frac{1}{\delta_L(z)} \tag{B25}$$

and the value of $1/\delta_L(z)$ is given in (B10).

The value $\delta_L^{-1}(z)$ is also finite at the limit $\delta \to 0$:

$$\lim_{\delta\to 0}\frac{1}{\delta_L(z)} \;=\; \frac{1}{6}\left[\ln\left(\frac{1}{z}\right)\right]^2 - \frac{1}{2}\left(p(\nu)^2 - p'(\nu)\right) - \frac{1}{3\ln(1/z)}\left(p(\nu)^3 - 3p'(\nu)p(\nu) + p''(\nu)\right)\;, \tag{B26}$$

where the Ψ''-function is the second derivation of the Ψ-function, $p''(\nu_q) \approx 251/108$ and $p''(\nu_G) \approx 2035/865$ are coming from quark counting rules.

(c) Bessel-like case. Following to the subsection (1c), we obtain:

(c1) in the $n \geq 1$ case:

$$J_{\delta,I}^{(1)}(n,z) \;=\; \frac{1}{(n+\delta)^2}\; f_I(z) + \mathcal{O}\left(\sqrt{\frac{\hat{d}}{\ln(1/z)}}\right)\;, \tag{B27}$$

(c2) in the $n = 0$ case:

$$J_{\delta,I}^{(1)}(0,z) \;=\; \frac{1}{\delta_I^2(z)}\; f_I(z) + \mathcal{O}(z^{1-\delta})\;, \tag{B28}$$

where

$$\frac{1}{\delta_I^2(z)} \;=\; -\frac{d}{d\delta}\frac{1}{\delta_I(z)} \tag{B29}$$

and the value of $\dfrac{1}{\delta_I(z)}$ is given in (B15).

The value $\delta_I^{-2}(z)$ is also finite at the limit $\delta \to 0$:

$$\lim_{\delta\to 0}\frac{1}{\delta_I^2(z)} \;=\; \frac{\ln(1/z)}{\hat{d}}\,\frac{\widetilde{I}_{k+2}\left(2\sqrt{\hat{d}\ln(1/z)}\right)}{\widetilde{I}_{k}\left(2\sqrt{\hat{d}\ln(1/z)}\right)} \;\approx\; \left(\lim_{\delta\to 0}\frac{1}{\delta_I(z)} - \frac{1}{4\hat{d}}\right)^2 + \frac{3(k+1)}{8\hat{d}^2} \tag{B30}$$

Note that the r.h.s. of (B16) is obtained from the expansion of the modified Bessel functions at $z \to 0$.

Note that we can represented the Eqs. (B20), (B24) and (B28) formally as follows

$$\delta^{-2}\; f_B(z) \;=\; \frac{1}{\delta_B^2(z)}\; f_B(z) \qquad (B = R, L, I), \tag{B31}$$

which has been used in Section V.

(3) Consider the Mellin integral

$$I_\delta(z) \;=\; \widetilde{K}(z) \otimes f(z) \equiv \int_z^1 \frac{\mathrm{d}y}{y}\; \hat{K}(y)\; f\left(\frac{z}{y}\right)$$

and define the moments of the kernel $\widetilde{K}(y)$ in the following form

$$K_n \;=\; \int_0^1 \mathrm{d}y\; y^{n-2}\; \widetilde{K}(y)\;.$$

In analogy with part (1) we have for the Regge-like case:

$$\begin{aligned} I_{\delta,R}(z) &= z^{-\delta}\int_z^1 \mathrm{d}y\; y^{\delta-1}\; \widetilde{K}(y)\;\left[\widetilde{f}(0) + \frac{z}{y}\;\widetilde{f}^{(1)}(0) + \ldots + \frac{1}{k!}\left(\frac{z}{y}\right)^k \widetilde{f}^{(k)}(0) + \ldots\right] \\ &= z^{-\delta}\left[K_{1+\delta}\;\widetilde{f}(0) + \mathcal{O}(z)\right] \\ &- \left[N_{1+\delta}(x)\widetilde{f}(0) + N_\delta(z)\widetilde{f}^{(1)}(0) + \ldots + \frac{1}{k!}\;N_{1+\delta-k}(z)\widetilde{f}^{(k)}(0) + \ldots\right]\;, \end{aligned} \tag{B32}$$

where

$$N_\eta(z) \ = \ \int_0^1 \mathrm{d}y \ y^{\eta-2} \ \widetilde{K}(zy) \ .$$

The case $K_{1+\delta} = 1/(n+\delta)$ corresponds to $\widetilde{K}(y) = y^n$ and has been already considered in part (1). In the more general cases (for example, $K_{1+\delta} = \Psi(1+\delta)+\gamma$) we can represent the "moment" $K_{1+\delta}$ as a series of the sort $\sum_{m=1} 1/(n+\delta+m)$.

So, for the initial integral at small x we get the simple equation:

$$I_{\delta,R}(z) \ = \ z^{-\delta} \ K_{R,1+\delta} \ \widetilde{f}(z) + \mathcal{O}(z^{1-\delta}) \ = \ K_{1+\delta} \ f_R(z) + \mathcal{O}(z^{1-\delta}) \ , \tag{B33}$$

where the coefficient $K_{R,1+\delta}$ coincides with the one $K_{1+\delta}$ in the case if K_n does not contain the term $1/(n-1)$. The coefficient $K_{R,1+\delta}$ contain the term $\delta_R^{-1}(z)$ if the term $1/(n-1)$ contributed to K_n. So, the function $K_{R,1+\delta}$ can be represented in the form:

$$K_{R,1+\delta} \ = \ K_{1+\delta_R(z)} \ . \tag{B34}$$

Repeating the analysis of the subparts *(b)* and *(c)*, one easily obtains

$$I_{\delta,L}(n,z) = K_{L,1+\delta} \ f_L(z) + \mathcal{O}\left(\frac{1}{\ln(1/z)}\right) \tag{B35}$$

$$I_{\delta,I}(n,z) = K_{I,1+\delta} \ f_I(z) + \mathcal{O}\left(\sqrt{\frac{\hat{d}}{\ln(1/z)}}\right) \ , \tag{B36}$$

where

$$K_{L,1+\delta} = K_{1+\delta_L(z)} \ , \tag{B37}$$

$$K_{I,1+\delta} = K_{1+\delta_I(z)} \ . \tag{B38}$$

Thus, in the non-singular case (i.e. in the case when K_n does not contain the term $1/(n-1)$) the results of tans formation of the Mellin convolution to usual products depend only on the δ value but not on the concrete shape of parton distribution. The presence of the term $1/(n-1)$ in K_n leads to the results depending on numerical value of δ. If δ is large (more precisely, if $z^{-\delta} \gg \mathrm{const}$), the presence of the term $1/(n-1)$ in K_n leads to the term $1/\delta$ in the functions $K_{i,1+\delta}$ $(i=R,L,I)$ (because the term z^{δ} is negligible in expressions for $1/\delta_i$) and the results do not also depend on the concrete shape of parton distribution. If δ is small (i.e., if $z^{-\delta} \approx 1+\delta\ln(1/z)$, that depends on concrete z values, of course), then the subasymptotic of parton distribution starts to play and the function $K_{i,1+\delta}$ $(i=R,L,I)$ contains the term $1/\delta_i$, which is determined by the both: asymptotics and subasymptotics of parton distributions.

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[129] The x dependence can also be obtained from the Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation [124–128], which is out of the scope of this work. However, in Section V, we use the twist-four anomalous dimensions from Refs. [46–49] obtained from BFKL results.
[130] Since we are only interested in the small x behavior and the initial conditions are given by the flat (x-independent) functions (see Eq. (15)), we use permanently the variable $z = x/x_0$ with values $0 < z < 1$ with some arbitrary $x_0 \leq 1$.
[131] Note that twist-four corrections are studied below in two approaches based on BFKL and DGLAP equations (see the Section V). However, we give here the results only for the DGLAP approach based on the infrared renormalon model because it contains a more complete calculation and the agreement with experimental data is much better.
[132] From now on, for a quantity $k(n)$ we use the notation $\hat{k}(n)$ for the singular part when $n \to 1$ and $\overline{k}(n)$ for the corresponding regular part.
[133] Hereafter we use the variables $\sigma_{\rm LO}$ and $\rho_{\rm LO}$, introduced in Refs. [35–37] for the case $Q^2 \geq Q_0^2$. In our article, they are generalized to arbitrary values of Q^2 and beyond the LO approximation (see below).
[134] The original results of [44] contain an error in the term $\bar{d}^q_{+-}(1)$, where the correct number 23 at $C_F = 4/3$ and $C_A = 3$ was mistakenly replaced by 134/3. With the wrong number the value of $\bar{d}^q_{+-}(1)$ was approximately 10 times higher then in the Table I. However, the results of fits do not depend practically on the mistake.
[135] This simplification is connected also with quite poor present knowledge about HT contributions in the BFKL-motivated approach.
[136] The projectors $\varepsilon^{\tau 4,\pm}_{ab}$ can be obtained from Eq. (10) in Ref. [44] with the replacement $d_\pm(n) \to d^{\tau 4}_\pm(n) = \hat{d}^{\tau 4}/(n-1) + \bar{d}^{\tau 4}(n)$.
[137] Contrary to Ref. [54, 55] we use here the variable $z = x/x_0$

TABLE II: The result of the LO and NLO fits to H1 (1996/97) [2] and ZEUS (1996/97) [7] data for different low Q^2 cuts. In the fits f is fixed to 4 flavors.

Q^2 [GeV^2] $\geq$	$A_G^{\tau 2}$	$A_q^{\tau 2}$	Q_0^2 [GeV^2]	$\chi^2/n.o.p.$
LO(H1 96/97 [2])				
1.5	0.797±.022	0.791±.026	0.304±.005	181/101
2.0	0.819±.022	0.781±.026	0.309±.005	139/98
2.5	0.869±.024	0.754±.027	0.319±.005	88/90
3.5	0.920±.028	0.733±.029	0.332±.006	61/81
LO(ZEUS 96/97 [7])				
2.7	0.918±.031	0.754±.040	0.317±.005	80/116
3.5	0.893±.034	0.780±.042	0.315±.006	76/111
NLO(H1 96/97 [2])				
1.5	−.013±.015	0.893±.028	0.494±.009	201/101
2.0	0.003±.015	0.882±.028	0.505±.009	153/98
2.5	0.042±.017	0.850±.029	0.526±.010	95/90
3.5	0.082±.020	0.824±.032	0.554±.012	63/81
NLO(ZEUS 96/97 [7])				
2.7	0.061±.023	0.844±.044	0.523±.011	82/116
3.5	0.044±.025	0.871±.046	0.520±.012	78/111
NLO(H1[2] + ZEUS[7])				
1.5 ($r_{\rm Z}$ = .963)	0.010±.013	0.873±.024	0.506±.007	286/217 [204/101, 82/116]
2.0 ($r_{\rm Z}$ = .964)	0.021±.013	0.864±.024	0.512±.007	233/214 [154/98, 79/116]
2.5 ($r_{\rm Z}$ = .963)	0.046±.013	0.839±.024	0.524±.008	171/206 [95/90, 76/116]
3.5 ($r_{\rm Z}$ = .962)	0.063±.015	0.829±.026	0.537±.008	140/192 [66/81, 74/111]

TABLE III: The result of the LO and NLO fits to H1 [2–6] and ZEUS [7–14] data for different low Q^2 cuts and different f.

Q^2 [GeV^2] $\geq$	$A_G^{\tau 2}$	$A_q^{\tau 2}$	Q_0^2 [GeV^2]	$\chi^2/n.o.p.$
LO ($f = 3$)				
0.5 ($r_{\rm H1}$ = .933, $r_{\rm Z}$ = .955)	1.216±.015	1.153±.015	0.306±.003	1163/667 [488/292, 675/375]
1.0 ($r_{\rm H1}$ = .939, $r_{\rm Z}$ = .966)	1.424±.023	0.977±.023	0.313±.003	854/631 [389/279, 465/352]
1.5 ($r_{\rm H1}$ = .946, $r_{\rm Z}$ = .969)	1.472±.024	0.950±.023	0.317±.003	775/614 [348/267, 427/347]
2.0 ($r_{\rm H1}$ = .953, $r_{\rm Z}$ = .971)	1.527±.025	0.923±.023	0.323±.003	673/591 [273/252, 400/339]
2.5 ($r_{\rm H1}$ = .958, $r_{\rm Z}$ = .971)	1.589±.026	0.890±.024	0.330±.003	580/573 [193/236, 387/337]
3.5 ($r_{\rm H1}$ = .963, $r_{\rm Z}$ = .971)	1.655±.030	0.866±.026	0.339±.004	501/532 [142/210, 359/322]
LO ($f = 4$)				
0.5 ($r_{\rm H1}$ = .934, $r_{\rm Z}$ = .957)	0.641±.010	0.937±.012	0.295±.003	1090/667 [455/292, 635/375]
1.0 ($r_{\rm H1}$ = .940, $r_{\rm Z}$ = .966)	0.755±.015	0.821±.019	0.301±.003	826/631 [373/279, 453/352]
1.5 ($r_{\rm H1}$ = .947, $r_{\rm Z}$ = .969)	0.784±.016	0.801±.019	0.304±.003	754/614 [335/267, 419/347]
2.0 ($r_{\rm H1}$ = .953, $r_{\rm Z}$ = .971)	0.817±.017	0.780±.019	0.310±.003	659/591 [264/252, 395/339]
2.5 ($r_{\rm H1}$ = .958, $r_{\rm Z}$ = .971)	0.855±.017	0.754±.020	0.316±.003	570/573 [188/236, 382/337]
3.5 ($r_{\rm H1}$ = .963, $r_{\rm Z}$ = .971)	0.892±.020	0.737±.021	0.325±.004	495/532 [140/210, 355/322]
NLO ($f = 3$)				
0.5 ($r_{\rm H1}$ = .929, $r_{\rm Z}$ = .951)	−.094±.009	1.358±.015	0.515±.006	1406/667 [599/292, 807/375]
1.0 ($r_{\rm H1}$ = .936, $r_{\rm Z}$ = .965)	0.072±.014	1.114±.024	0.526±.006	966/631 [455/279, 511/352]
1.5 ($r_{\rm H1}$ = .944, $r_{\rm Z}$ = .968)	0.109±.015	1.078±.025	0.535±.006	863/614 [403/267, 460/347]
2.0 ($r_{\rm H1}$ = .952, $r_{\rm Z}$ = .971)	0.151±.016	1.045±.025	0.548±.006	735/591 [311/252, 424/339]
2.5 ($r_{\rm H1}$ = .958, $r_{\rm Z}$ = .970)	0.198±.016	1.006±.025	0.564±.006	620/573 [213/236, 407/337]
3.5 ($r_{\rm H1}$ = .963, $r_{\rm Z}$ = .971)	0.254±.019	0.972±.027	0.587±.007	523/532 [151/210, 372/322]
NLO ($f = 4$)				
0.5 ($r_{\rm H1}$ = .932, $r_{\rm Z}$ = .955)	−.142±.006	1.087±.012	0.478±.006	1229/667 [514/292, 715/375]
1.0 ($r_{\rm H1}$ = .938, $r_{\rm Z}$ = .966)	−.042±.011	0.929±.021	0.487±.006	884/631 [407/279, 477/352]
1.5 ($r_{\rm H1}$ = .946, $r_{\rm Z}$ = .969)	−.020±.011	0.903±.021	0.495±.006	798/614 [363/267, 435/347]
2.0 ($r_{\rm H1}$ = .953, $r_{\rm Z}$ = .971)	0.006±.012	0.877±.021	0.506±.006	688/591 [282/252, 406/339]
2.5 ($r_{\rm H1}$ = .958, $r_{\rm Z}$ = .971)	0.035±.012	0.847±.022	0.520±.006	589/573 [197/236, 392/337]
3.5 ($r_{\rm H1}$ = .963, $r_{\rm Z}$ = .972)	0.065±.014	0.826±.023	0.539±.007	505/532 [143/210, 362/322]

TABLE IV: The result of the LO and NLO fits to H1 (1996/97) [2] data. Power corrections included for different values of the parameter b and in the infrared renormalon case.

H1 96/97 [2]	$A_G^{\tau 2}$	$A_q^{\tau 2}$	$A_G^{\tau 4}$ ($a_G^{\tau 4}$) ($a_G^{\tau 6}$)	$A_q^{\tau 4}$ ($a_q^{\tau 4}$) ($a_q^{\tau 6}$)	Q_0^2 [GeV^2]	$\chi^2/n.o.p.$
LO ($f=4$)						
no $h\tau$	0.797±.022	0.791±.026	—	—	0.304±.005	181/101
$b=0$	1.214±.060	0.426±.054	—	0.969±.127	0.360±.009	124/101
$b=1$	1.263±.070	0.436±.051	−.496±.062	1.127±.142	0.388±.022	119/101
$b=a^2/2$	1.321±.072	0.446±.049	−.523±.065	1.205±.148	0.417±.023	106/101
$R\tau 4$	1.155±.060	0.582±.032	−.310±.171	0.230±.078	0.381±.020	56/101
			(0.000 fix)	(0.000 fix)		
renorm.	1.037±.121	0.668±.073	−.011±.259	−.007±.122	0.356±.035	54/101
			−.486±.841	0.084±.325		
NLO ($f=4$)						
no $h\tau$	−.013±.015	0.893±.028	—	—	0.494±.009	201/101
$b=0$	−.024±.017	0.882±.029	—	−.001±.000	0.473±.017	199/101
$b=1$	0.316±.047	0.474±.056	−.542±.065	1.219±.147	0.600±.030	133/101
$b=a^2/2$	0.336±.045	0.492±.053	−.603±.067	1.362±.152	0.635±.030	127/101
$R\tau 4$	0.144±.078	0.764±.056	−.692±.275	0.155±.021	0.576±.060	55/101
			(0.000 fix)	(0.000 fix)		
renorm.	0.102±.086	0.800±.066	−1.327±1.218	0.310±.281	0.548±.067	54/101
			0.412±.834	0.063±.144		

TABLE V: The result of the LO and NLO fits to ZEUS (1996/97) [7] data. Power corrections included for different values of the parameter b and in the infrared renormalon case.

ZEUS 96/97 [7]	$A_G^{\tau 2}$	$A_q^{\tau 2}$	$A_G^{\tau 4}$ ($a_G^{\tau 4}$) ($a_G^{\tau 6}$)	$A_q^{\tau 4}$ ($a_q^{\tau 4}$) ($a_q^{\tau 6}$)	Q_0^2 [GeV^2]	$\chi^2/n.o.p.$
LO ($f=4$)						
no $h\tau$	0.918±.031	0.754±.040	—	—	0.317±.005	80/116
$b=0$	0.891±.067	0.780±.070	—	−.093±.203	0.314±.009	80/116
$b=1$	0.910±.074	0.780±.068	0.046±.101	−.101±.229	0.324±.023	79/116
$b=a^2/2$	0.920±.069	0.786±.066	0.083±.117	−.179±.263	0.330±.019	78/116
$R\tau 4$	0.980±.063	0.739±.050	0.344±.329	−.137±.135	0.343±.021	78/116
			(0.000 fix)	(0.000 fix)		
renorm.	0.859±.087	0.757±.074	−2.439±1.207	1.014±.559	0.281±.024	68/116
			−10.66±3.60	4.99±1.78		
NLO ($f=4$)						
no $h\tau$	0.061±.023	0.844±.044	—	—	0.523±.011	82/116
$b=0$	0.067±.030	0.849±.046	—	−.001±.002	0.533±.034	81/116
$b=1$	0.062±.015	0.859±.026	0.020±.002	−.044±.005	0.534±.017	81/116
$b=a^2/2$	0.071±.055	0.866±.073	0.046±.122	−.101±.275	0.549±.037	80/116
$R\tau 4$	0.083±.081	0.823±.078	−.046±.313	0.016±.041	0.533±.054	81/116
			(0.000 fix)	(0.000 fix)		
renorm.	−.329±.068	1.242±.094	−1.599±.643	−.177±.173	0.312±.027	64/116
			−16.008±2.451	2.253±0.492		

TABLE VI: The result of the LO and NLO fits to H1 [2–6] and ZEUS [7–14] data at $Q^2 \geq 1.5$ GeV^2. Power corrections included for different values of the parameter b and in the infrared renormalon case.

H1[2–6] + ZEUS[7–14] $Q^2 \geq 1.5$ GeV^2	$A_G^{\tau 2}$	$A_q^{\tau 2}$	$A_G^{\tau 4}$ ($a_G^{\tau 4}$) ($a_G^{\tau 6}$)	$A_q^{\tau 4}$ ($a_q^{\tau 4}$) ($a_q^{\tau 6}$)	Q_0^2 [GeV^2]	$\chi^2/n.o.p.$
LO ($f=3$)						
no $h\tau$ ($r_{\rm H1}=.946$, $r_{\rm Z}=.969$)	1.472±.024	0.950±.023	—	—	0.317±.003	775/614 [348/267, 427/347]
$b=0$ ($r_{\rm H1}=.953$, $r_{\rm Z}=.970$)	2.083±.056	0.513±.042	—	1.275±.103	0.372±.006	628/614 [246/267, 382/347]
$b=1$ ($r_{\rm H1}=.954$, $r_{\rm Z}=.971$)	2.164±.068	0.528±.040	−.623±.049	1.422±.113	0.401±.014	616/614 [240/267, 376/347]
$b=a^2/2$ ($r_{\rm H1}=.954$, $r_{\rm Z}=.972$)	2.224±.067	0.546±.039	−.617±.051	1.431±.116	0.421±.013	591/614 [224/267, 367/347]
$R\tau 4$	2.012±.062	0.687±.029	−.279±.135	0.326±.088	0.390±.012	503/614
($r_{\rm H1}=.959$, $r_{\rm Z}=.973$)			(0.000 fix)	(0.000 fix)		[151/267, 352/347]
renorm.	1.826±.100	0.784±.050	−.064±.185	0.006±.092	0.360±.019	498/614
($r_{\rm H1}=.959$, $r_{\rm Z}=.972$)			−1.245±.718	.525±.383		[149/267, 349/347]
LO ($f=4$)						
no $h\tau$ ($r_{\rm H1}=.947$, $r_{\rm Z}=.969$)	0.784±.016	0.801±.019	—	—	0.304±.003	754/614 [335/267, 419/347]
$b=0$ ($r_{\rm H1}=.953$, $r_{\rm Z}=.970$)	1.157±.036	0.461±.035	—	0.950±.082	0.353±.005	625/614 [244/267, 381/347]
$b=1$ ($r_{\rm H1}=.954$, $r_{\rm Z}=.971$)	1.202±.042	0.477±.033	−.478±.040	1.088±.091	0.383±.014	612/614 [237/267, 375/347]
$b=a^2/2$ ($r_{\rm H1}=.954$, $r_{\rm Z}=.972$)	1.232±.041	0.494±.032	−.481±.042	1.107±.096	0.402±.013	586/614 [220/267, 366/347]
$R\tau 4$	1.125±.037	0.582±.024	−.172±.100	0.162±.047	0.376±.012	505/614
($r_{\rm H1}=.959$, $r_{\rm Z}=.973$)			(0.000 fix)	(0.000 fix)		[153/267, 352/347]
renorm.	0.990±.060	0.679±.043	−.009±.161	−.019±.092	0.345±.017	497/614
($r_{\rm H1}=.959$, $r_{\rm Z}=.972$)			−.980±.497	0.276±.196		[149/267, 348/347]
NLO ($f=3$)						
no $h\tau$ ($r_{\rm H1}=.944$, $r_{\rm Z}=.968$)	0.109±.015	1.078±.025	—	—	0.535±.006	863/614 [403/267, 460/347]
$b=0$ ($r_{\rm H1}=.945$, $r_{\rm Z}=.968$)	0.101±.018	1.073±.025	—	−.001±.001	0.527±.011	862/614 [401/267, 461/347]
$b=1$ ($r_{\rm H1}=.953$, $r_{\rm Z}=.970$)	0.600±.043	0.563±.043	−.735±.052	1.655±.117	0.661±.020	669/614 [273/267, 396/347]
$b=a^2/2$ ($r_{\rm H1}=.953$, $r_{\rm Z}=.970$)	0.630±.042	0.585±.041	−.783±.053	1.771±.120	0.691±.019	654/614 [265/267, 389/347]
$R\tau 4$	0.410±.067	0.864±.041	−.660±.185	0.265±.024	0.643±.034	505/614
($r_{\rm H1}=.960$, $r_{\rm Z}=.973$)			(0.000 fix)	(0.000 fix)		[151/267, 354/347]
renorm.	0.188±.080	0.985±.045	−3.081±1.019	.957±.315	0.530±.041	498/614
($r_{\rm H1}=.959$, $r_{\rm Z}=.972$)			.036±.422	.437±.164		[149/267, 349/347]
NLO ($f=4$)						
no $h\tau$ ($r_{\rm H1}=.946$, $r_{\rm Z}=.969$)	−.020±.011	0.903±.021	—	—	0.495±.006	798/614 [363/267, 435/347]
$b=0$ ($r_{\rm H1}=.946$, $r_{\rm Z}=.969$)	−.024±.013	0.899±.022	—	0.00±.0004	0.488±.011	798/614 [362/267, 436/347]
$b=1$ ($r_{\rm H1}=.953$, $r_{\rm Z}=.970$)	0.288±.029	0.515±.037	−.535±.042	1.205±.095	0.602±.019	645/614 [256/267, 389/347]
$b=a^2/2$ ($r_{\rm H1}=.954$, $r_{\rm Z}=.971$)	0.301±.028	0.535±.035	−.580±.044	1.311±.100	0.631±.019	629/614 [248/267, 381/347]
$R\tau 4$	0.156±.041	0.734±.035	−.522±.153	0.141±.014	0.579±.031	506/614
($r_{\rm H1}=.960$, $r_{\rm Z}=.973$)			(0.000 fix)	(0.000 fix)		[151/267, 355/347]
renorm.	0.041±.045	0.824±.034	−2.765±.968	.676±.240	0.493±.037	500/614
($r_{\rm H1}=.959$, $r_{\rm Z}=.973$)			.939±.718	.252±.099		[151/267, 349/347]

TABLE VII: The result of the LO and NLO fits to H1 [2–6] and ZEUS [7–14] at $Q^2 \geq 0.5$ GeV^2. Power corrections included in the infrared renormalon case.

H1[2–6] + ZEUS[7–14] $Q^2 \geq 0.5$ GeV^2	$A_G^{\tau 2}$	$A_q^{\tau 2}$	$\mathrm{a}_G^{\tau 4}$ $\mathrm{a}_G^{\tau 6}$	$\mathrm{a}_q^{\tau 4}$ $\mathrm{a}_q^{\tau 6}$	Q_0^2 [GeV^2]	$\chi^2/n.o.p.$
LO $R\tau 4$ ($f=3$)	2.212±.050	0.602±.027	0.238±.019	−.014±.005	0.428±.008	569/667
($r_{\mathrm{H1}} = .953$, $r_{\mathrm{Z}} = .975$)			(0.000 fix)	(0.000 fix)		[192/292, 377/375]
LO ($f=3$)	2.161±.055	0.633±.029	−.002±.024	0.165±.024	0.421±.010	553/667
($r_{\mathrm{H1}} = .955$, $r_{\mathrm{Z}} = .974$)			−.017±.017	0.053±.010		[181/292, 372/375]
LO $R\tau 4$ ($f=4$)	1.234±.031	0.518±.023	0.201±.016	−.011±.003	0.407±.008	573/667
($r_{\mathrm{H1}} = .953$, $r_{\mathrm{Z}} = .975$)			(0.000 fix)	(0.000 fix)		[193/292, 380/375]
LO ($f=4$)	1.211±.033	0.539±.023	−.002±.020	0.102±.015	0.404±.009	555/667
($r_{\mathrm{H1}} = .955$, $r_{\mathrm{Z}} = .974$)			0.001±.010	0.031±.005		[182/292, 373/375]
NLO $R\tau 4$ ($f=3$)	1.014±.057	0.521±.034	0.632±.048	0.188±.026	0.956±.031	621/667
($r_{\mathrm{H1}} = .951$, $r_{\mathrm{Z}} = .975$)			(0.000 fix)	(0.000 fix)		[220/292, 401/375]
NLO ($f=3$)	0.617±.058	0.742±.038	−.129±.102	0.224±.022	0.746±.030	562/667
($r_{\mathrm{H1}} = .956$, $r_{\mathrm{Z}} = .975$)			−.203±.053	0.061±.010		[182/292, 380/375]
NLO $R\tau 4$ ($f=4$)	0.485±.033	0.476±.029	0.556±.042	0.071±.008	0.826±.026	617/667
($r_{\mathrm{H1}} = .950$, $r_{\mathrm{Z}} = .975$)			(0.000 fix)	(0.000 fix)		[222/292, 395/375]
NLO ($f=4$)	0.279±.038	0.640±.034	−.143±.100	0.140±.015	0.672±.029	565/667
($r_{\mathrm{H1}} = .955$, $r_{\mathrm{Z}} = .974$)			−.044±.050	0.043±.007		[184/292, 381/375]

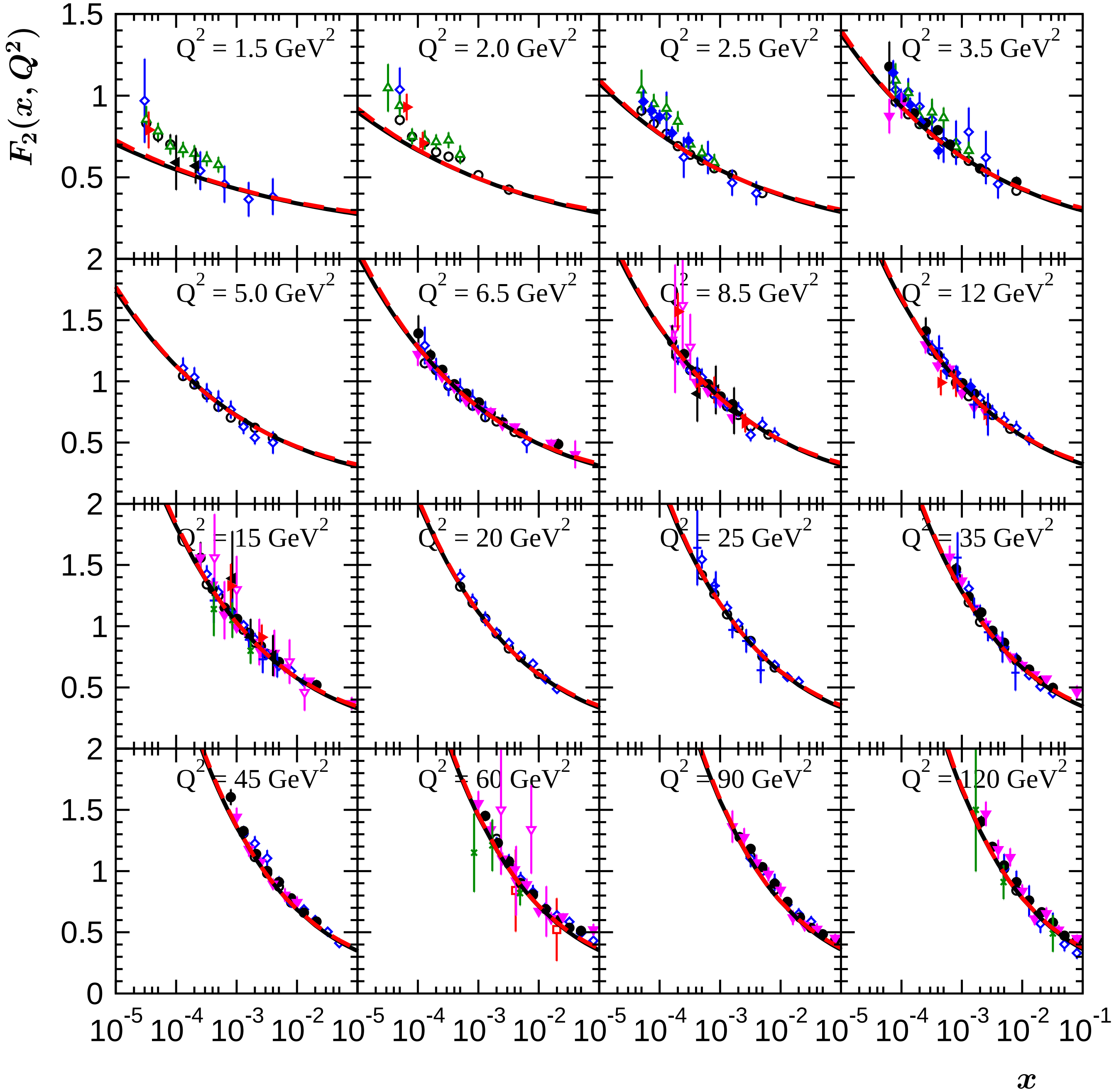

FIG. 1: $F_2^{\tau 2}(x, Q^2)$ as a function of x for different Q^2 bins. The experimental points are from H1[2–6] (open points) and ZEUS[7–14] (solid points). The solid, black line represents the NLO fit with χ^2/n.d.f. = 798/611 = 1.31 [$A_G^{\tau 2} = -.020$, $A_q^{\tau 2} = .903$, $Q_0^2 = .495$ GeV^2]. The long dashed, red line represents the LO fit with χ^2/n.d.f. = 754/611 = 1.23 [$A_G^{\tau 2} = .784$, $A_q^{\tau 2} = .801$, $Q_0^2 = .304$ GeV^2].

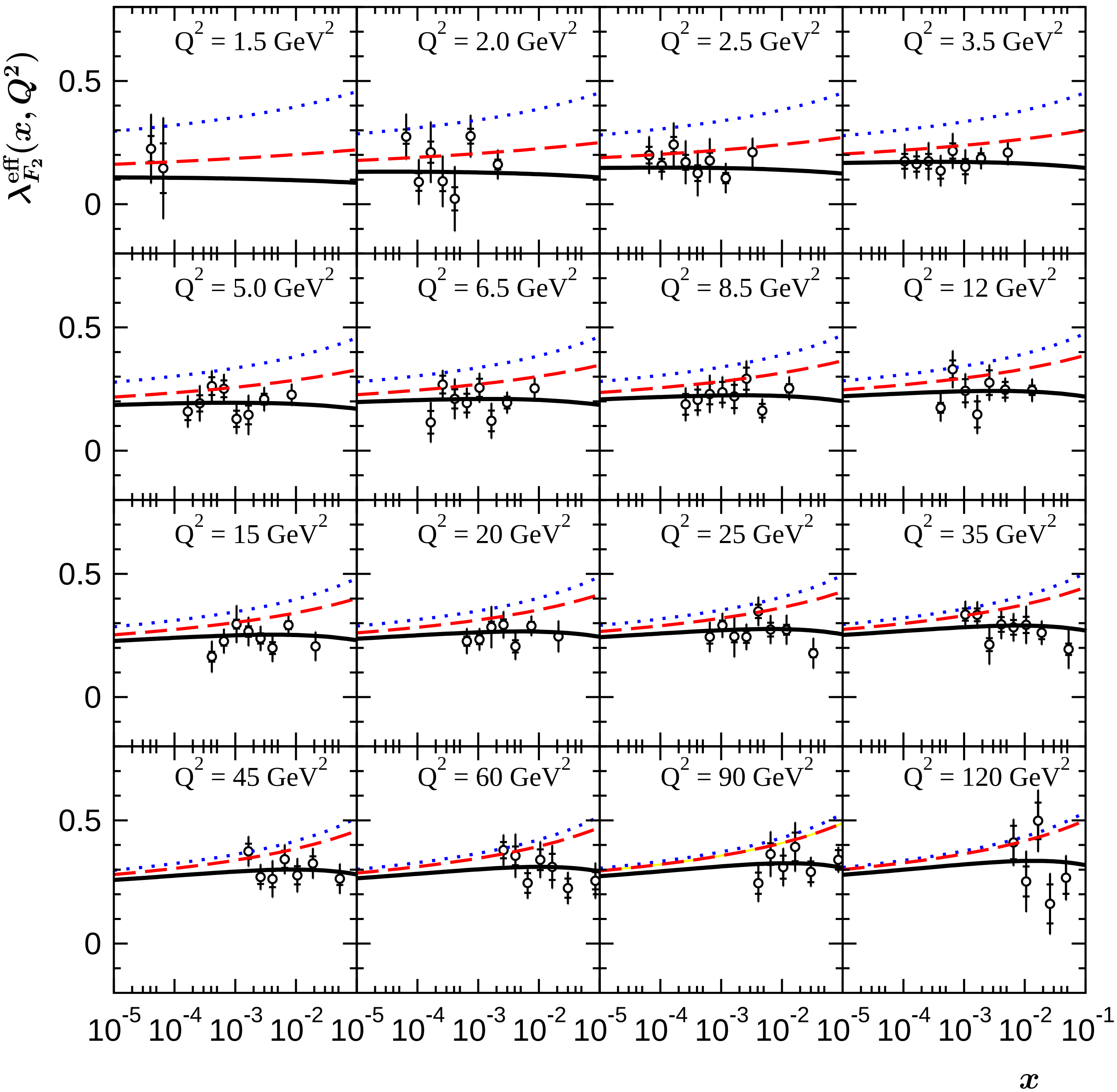

FIG. 2: The derivative function (effective slope) $\lambda^{\rm eff,\tau2}_{F_2}(x,Q^2) = \partial \ln F_2^{\tau2}(x,Q^2)/\partial \ln(1/x)$ as a function of x for different Q^2 bins. The experimental points are from H1[2]. The outer error bars include statistical and systematical errors added in quadrature, while the inner error bars correspond to statistical errors only. The solid, black line represents the NLO fit with $\chi^2/\text{n.d.f.} = 798/611 = 1.31$ $[A_G^{\tau2} = -.020,\ A_q^{\tau2} = .903,\ Q_0^2 = .495\ \text{GeV}^2]$, while the long dashed, red line (hardly distinguished from the solid one) is the LO fit with $\chi^2/\text{n.d.f.} = 754/611 = 1.23$ $[A_G^{\tau2} = .784,\ A_q^{\tau2} = .801,\ Q_0^2 = .304\ \text{GeV}^2]$. The dotted, blue line corresponds to the asymptotic expression $\lambda^{\rm eff,\tau2}_{F_2,\rm as}(x,Q^2)$ in Eq. (43c).

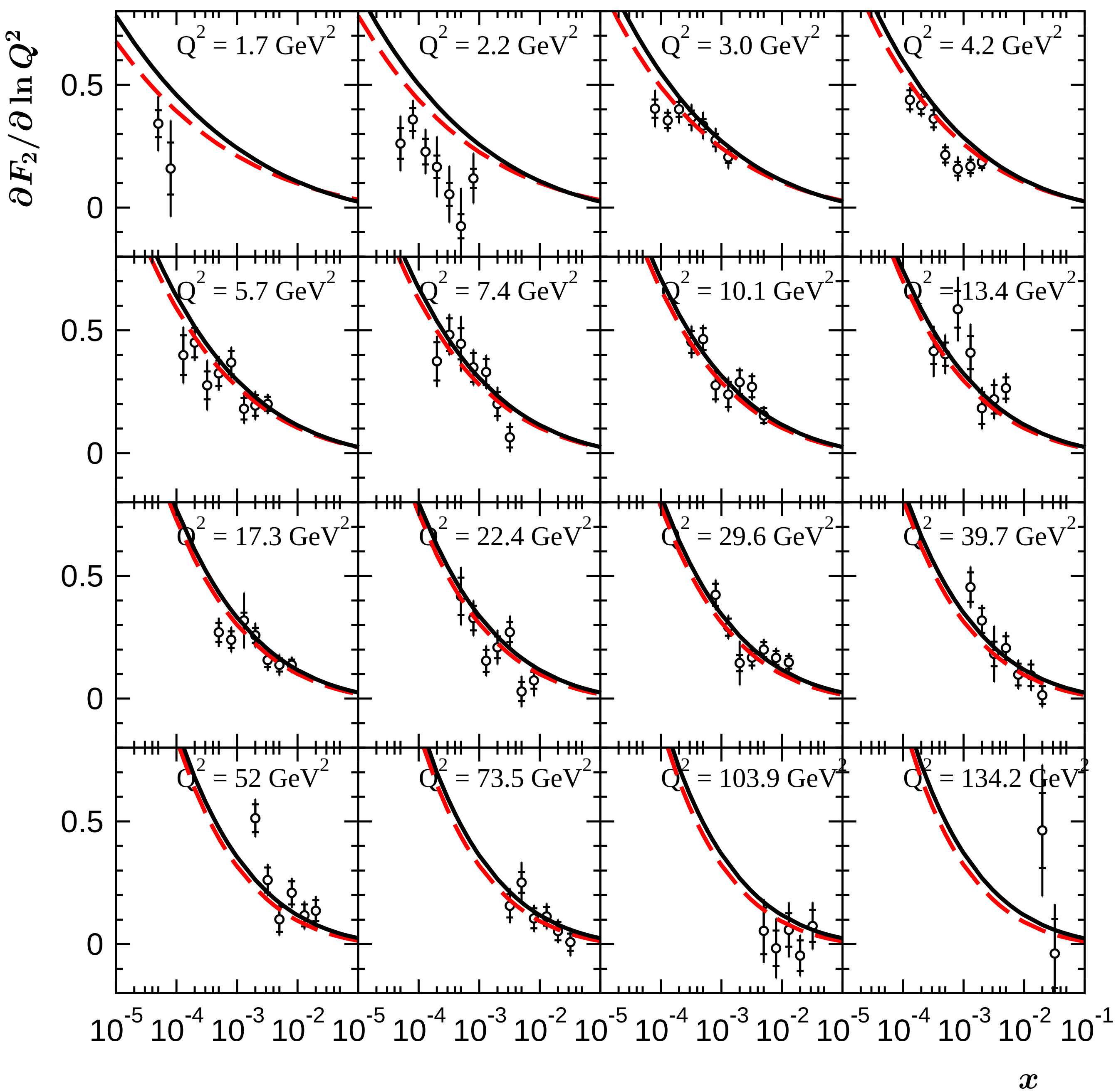

FIG. 3: The derivative function $\partial F_2^{\tau 2}(x, Q^2)/\partial \ln Q^2$ taken at fixed Q^2 and plotted as a function of x. The experimental points are from H1[2]. The outer error bars represent the quadratic sum of statistical and systematical errors. The inner error bars show the statistical error only. The solid, black line represents the NLO fit with χ^2/n.d.f. = 798/611 = 1.31 [$A_G^{\tau 2} = -.020$, $A_q^{\tau 2} = .903$, $Q_0^2 = .495$ GeV^2], while the long dashed, red line is the LO fit with χ^2/n.d.f. = 754/611 = 1.23 [$A_G^{\tau 2} = .784$, $A_q^{\tau 2} = .801$, $Q_0^2 = .304$ GeV^2].

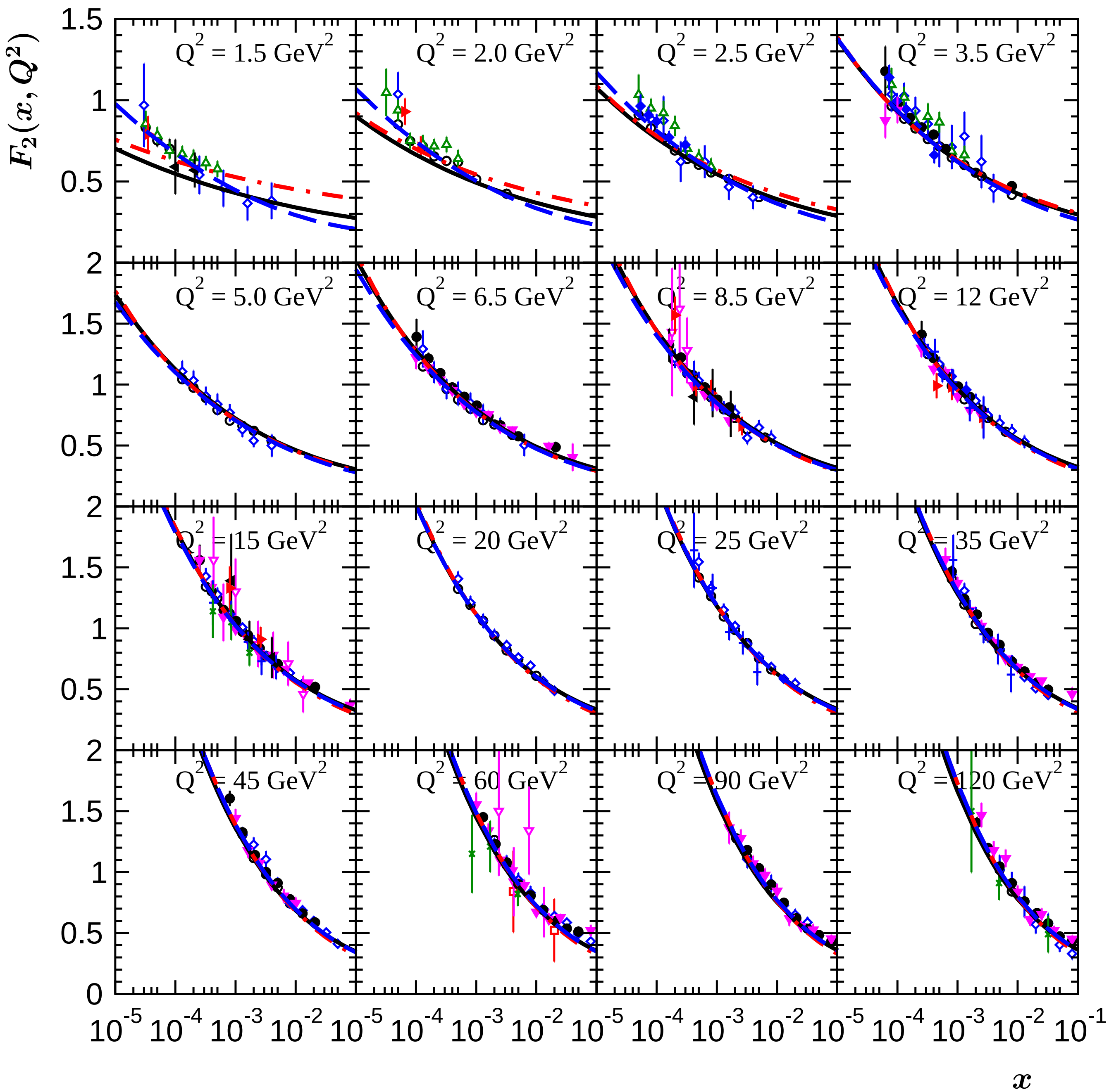

FIG. 4: $F_2(x, Q^2)$ as a function of x for different Q^2 bins. The experimental points are the same as on Figure 1. The solid, black line represents the NLO fit alone with $\chi^2/\text{n.d.f.} = 798/611 = 1.31$ [$A_G^{\tau 2} = -.020,\ A_q^{\tau 2} = .903,\ Q_0^2 = .495\ \text{GeV}^2$]. The dash-dotted, red curve represents the BFKL-motivated estimations for higher twist contribution to $F_2(x, Q^2)$ with the value of the parameter $b = a^2/2$. The corresponding $\chi^2/\text{n.d.f.} = 629/609 = 1.03$ [$A_G^{\tau 2} = .301,\ A_q^{\tau 2} = .535,\ Q_0^2 = .631\ \text{GeV}^2$ and $A_G^{\tau 4} = -.580\ \text{GeV}^2,\ A_q^{\tau 4} = 1.311\ \text{GeV}^2$]. The dashed, blue curve is obtained from the fits at the NLO, when the renormalon contributions of higher-twist terms have been incorporated. The corresponding $\chi^2/\text{n.d.f.} = 500/607 = 0.82$ [$A_G^{\tau 2} = .041,\ A_q^{\tau 2} = .824,\ Q_0^2 = .493\ \text{GeV}^2$ and $\text{a}_G^{\tau 4} = -2.765\ \text{GeV}^2,\ \text{a}_q^{\tau 4} = .676\ \text{GeV}^2,\ \text{a}_G^{\tau 6} = .939\ \text{GeV}^4,\ \text{a}_q^{\tau 6} = .252\ \text{GeV}^4$].

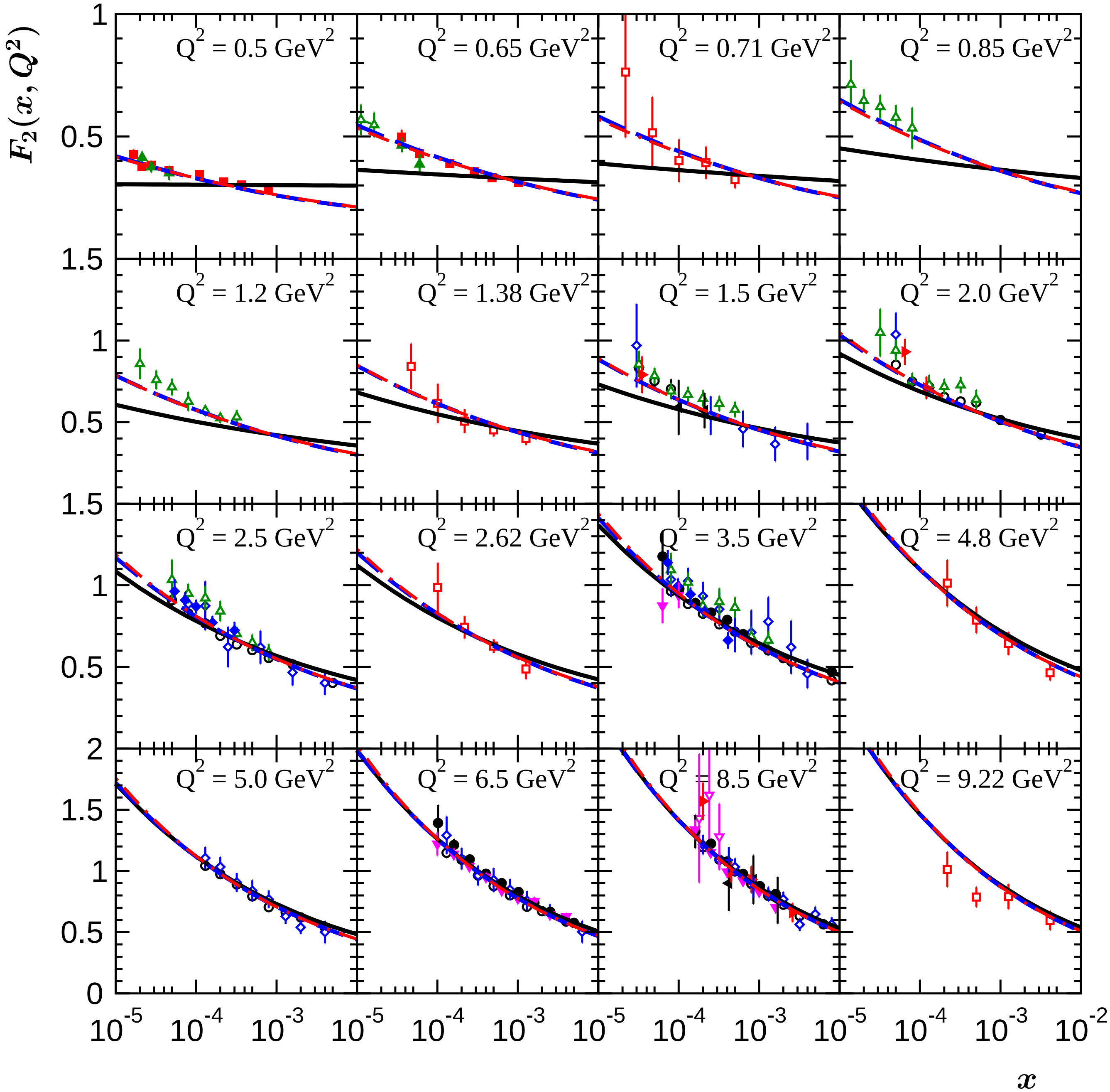

FIG. 5: $F_2(x,Q^2)$ as a function of x for different Q^2 bins. The experimental points are from H1[2–6] (open points) and ZEUS[7–14] (solid points). The solid, black line represents the NLO fit alone with χ^2/n.d.f. = 798/611 = 1.31 [$A_G^{\tau 2} = -.020$, $A_q^{\tau 2} = .903$, $Q_0^2 = .495$ GeV^2]. The dashed, blue curve is obtained from the fit at the NLO, when the renormalon contributions of higher-twist terms have been incorporated. The corresponding χ^2/n.d.f. = 565/660 = 0.86 [$A_G^{\tau 2} = .279$, $A_q^{\tau 2} = .640$, $Q_0^2 = .672$ GeV^2 and $a_G^{\tau 4} = -.143$ GeV^2, $a_q^{\tau 4} = .140$ GeV^2, $a_G^{\tau 6} = -.044$ GeV^4, $a_q^{\tau 6} = .043$ GeV^4]. The dash-dotted, red curve (hardly distinguished from the dashed one) represents the fit at the LO together with the renormalon contributions of higher-twist terms. The corresponding χ^2/n.d.f. = 555/660 = 0.84 [$A_G^{\tau 2} = 1.211$, $A_q^{\tau 2} = .539$, $Q_0^2 = .404$ GeV^2 and $a_G^{\tau 4} = -.002$ GeV^2, $a_q^{\tau 4} = .102$ GeV^2, $a_G^{\tau 6} = .001$ GeV^4, $a_q^{\tau 6} = .031$ GeV^4].

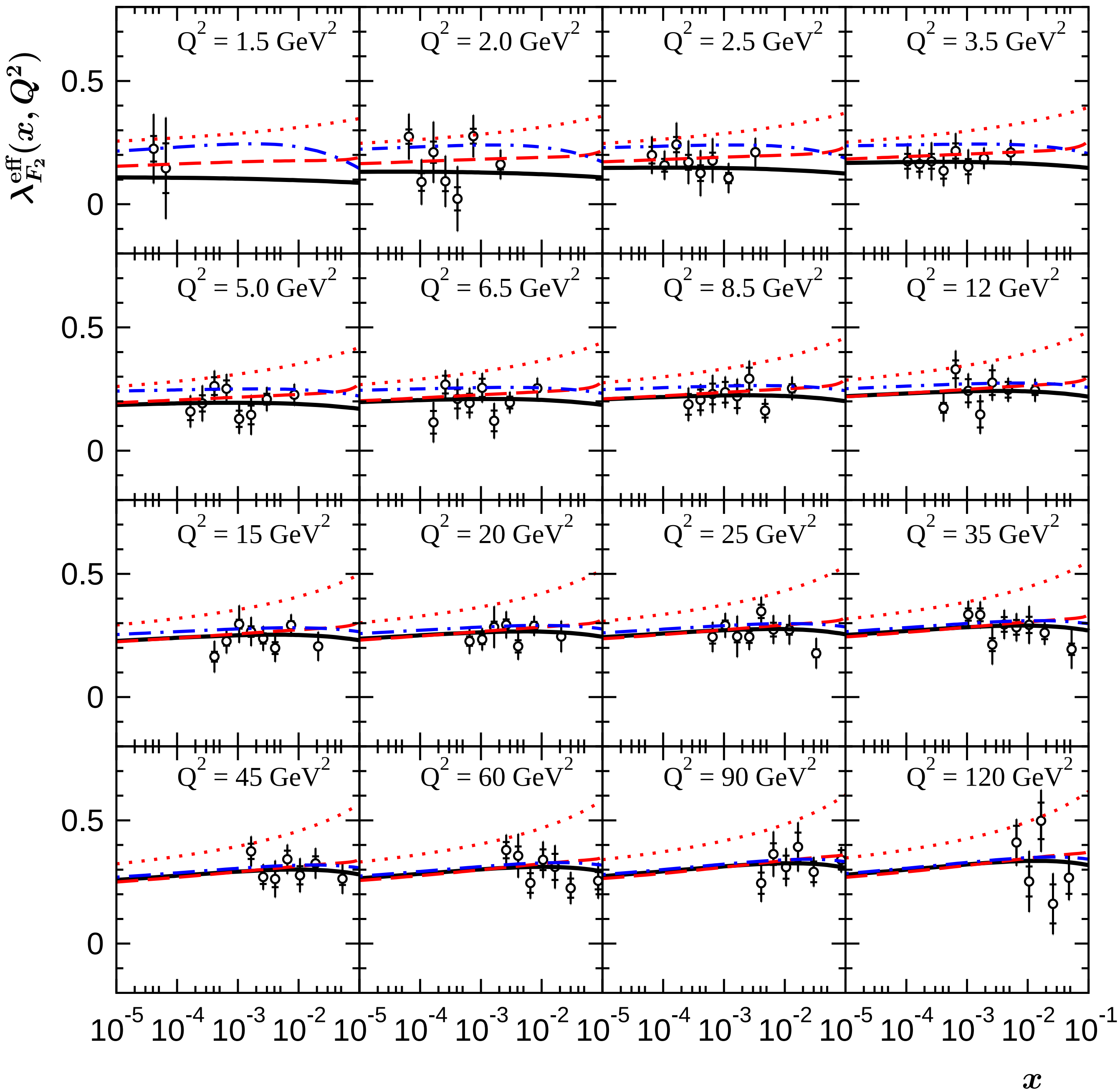

FIG. 6: The derivative function (effective slope) $\lambda^{\rm eff}_{F_2} = \partial \ln F_2(x,Q^2)/\partial \ln(1/x)$ as a function of x for different Q^2 bins. The experimental points and the solid, black line (NLO fit with χ^2/n.d.f. = 798/611 = 1.31) are the same as on Figure 2. All other curves are obtained from the fits, when the renormalon contributions of higher-twist terms have been incorporated. The dashed, red one is the same as on the Figure 5 with the corresponding χ^2/n.d.f. = 565/660 = 0.86, while the dash-dotted, blue line is the one from the Figure 4 with χ^2/n.d.f. = 500/607 = 0.82. The dotted, red line corresponds to the asymptotic LO expression $\lambda^{\rm eff}_{F_2,\rm as}(x,Q^2)$ in Eq. (79), plotted at χ^2/n.d.f. = 573/667 = 0.87 [$A^{\tau 2}_G = 1.234$, $A^{\tau 2}_q = .518$, $Q^2_0 = .407$ GeV^2 and $a^{\tau 4}_G = .201$ GeV^2, $a^{\tau 4}_q = -.011$ GeV^2].

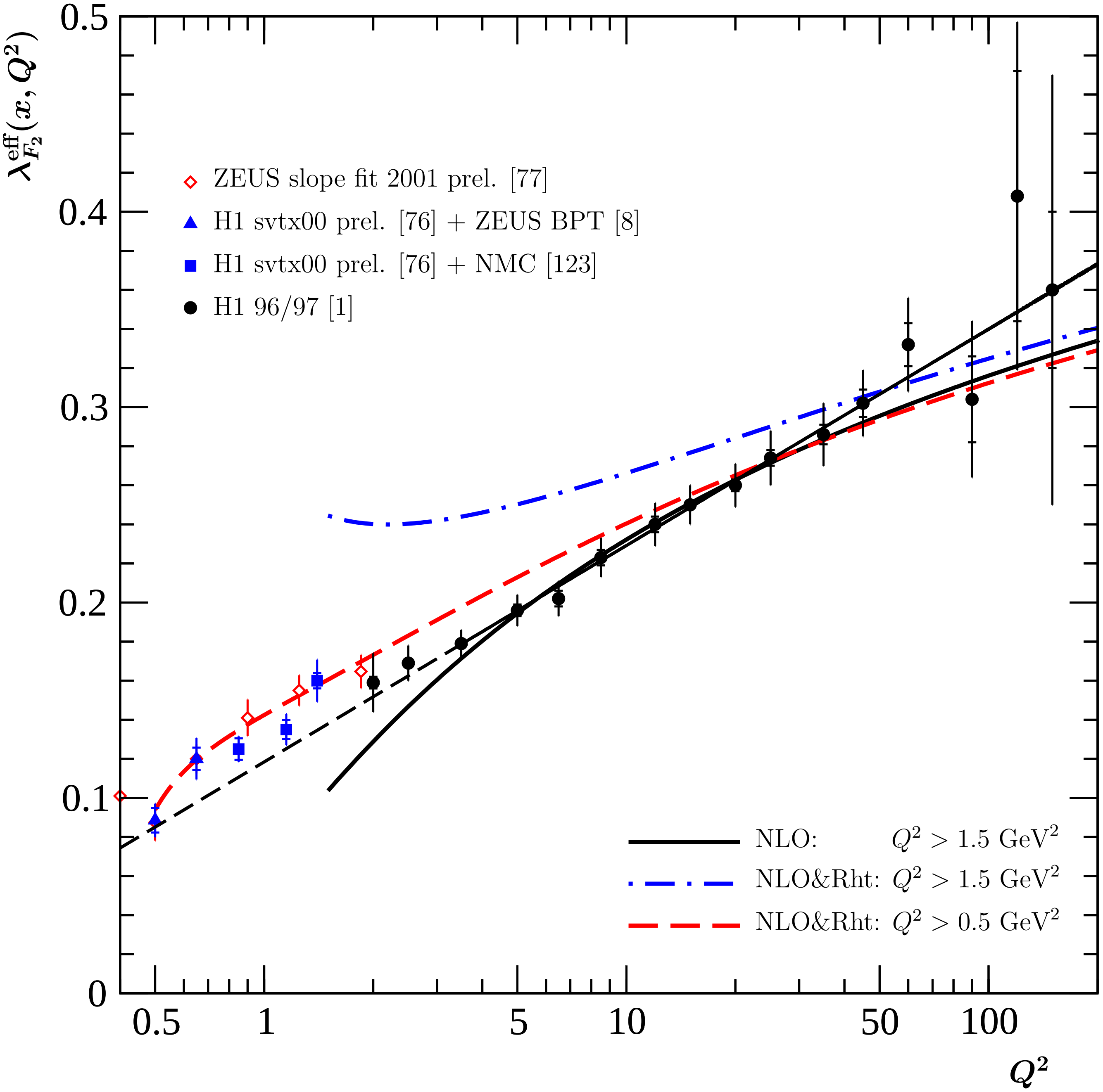

FIG. 7: The derivative function (effective slope) $\lambda^{\rm eff}_{F_2} = \partial \ln F_2(x,Q^2)/\partial \ln(1/x)$ as a function of Q^2. The experimental points are those H1 and ZEUS have fitted their $x \leq 0.01$ data to the form $F_2 = c(Q^2)x^{-\lambda(Q^2)}$: black points – H1 F_2 data [1]; blue squares – H1 data [76] combined with NMC data [123]; blue triangles – H1 data [76] combined with low Q^2 ZEUS BPT data [8]; open red diamonds – preliminary ZEUS slope fit 2001 [77]. The inner error bars illustarte the statistical uncertainties, the full error bars represent the statistical and systematic uncertanties added in quadrature. The data are compared with a parametrization [1] in which $\lambda(Q^2) = a\ln[Q^2/\Lambda^2]$ grows logarithmically with Q^2 [$a = .0481$, $\Lambda = 292$ MeV], using data for $Q^2 \geq 3.5$ GeV^2 (black solid straight line goes to approximated short-dashed one). The solid, black line (NLO fit with χ^2/n.d.f. = 798/611 = 1.31), the long-dashed, red one (NLO&Rht fit with χ^2/n.d.f. = 565/660 = 0.86) and the dash-dotted, blue one (NLO&Rht fit with χ^2/n.d.f. = 500/607 = 0.82) are the same as on the previous Figure 6. The value of x was fixed to 10^{-3} for all curves.

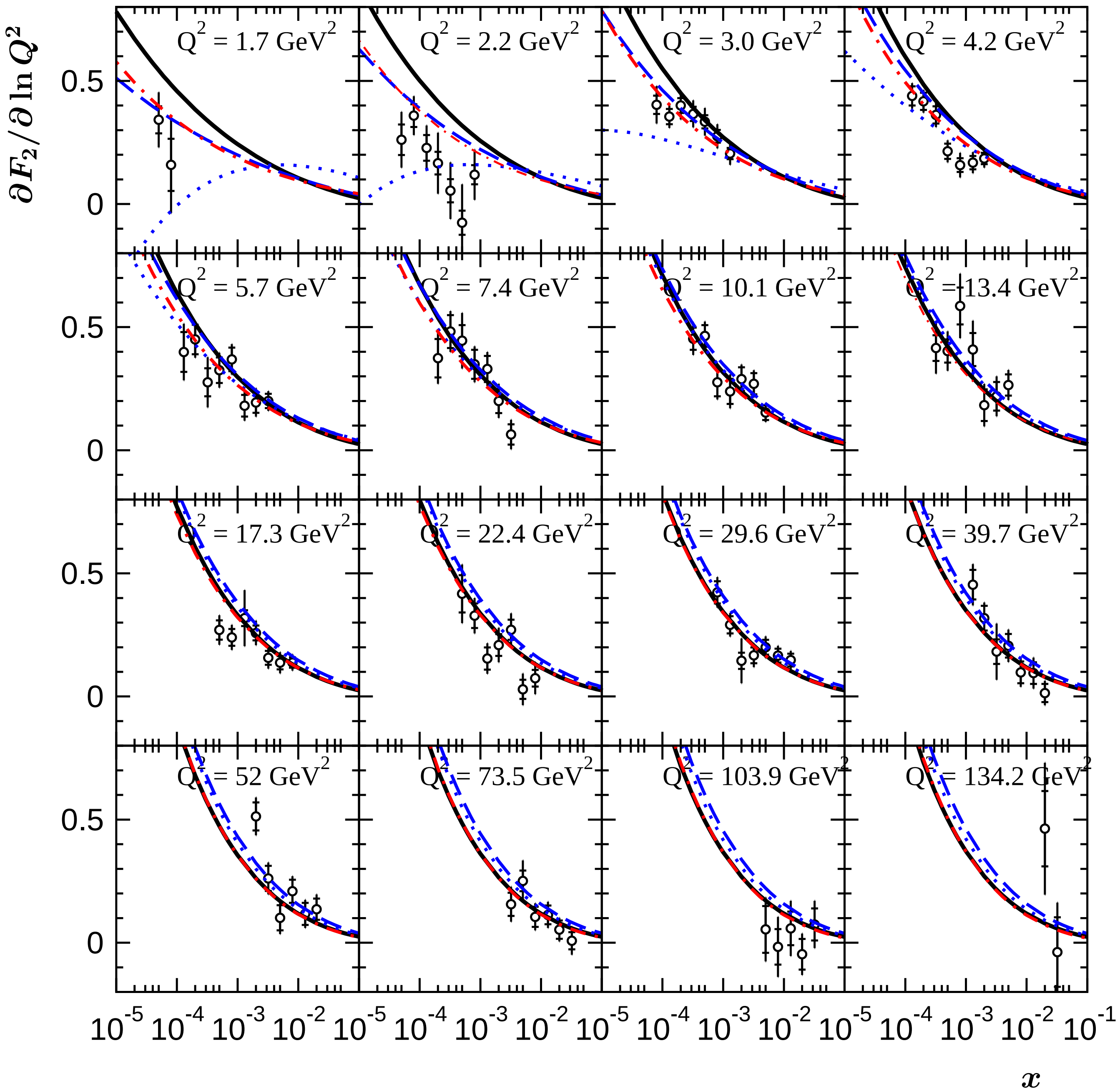

FIG. 8: The derivative function $\partial F_2(x,Q^2)/\partial \ln Q^2$ taken at fixed Q^2 and plotted as a function of x. The experimental points and the solid, black line (NLO fit with $\chi^2/\text{n.d.f.} = 798/611 = 1.31$) are the same as on Figure 3. The dashed and dotted, blue curves are obtained from the fit at the NLO, when the renormalon contributions of higher-twist terms have been incorporated. The dashed one is the same as on the Figure 5 with the corresponding $\chi^2/\text{n.d.f.} = 565/660 = 0.86$, while the dotted line is the one from the Figure 4 with $\chi^2/\text{n.d.f.} = 500/607 = 0.82$. The dash-dotted, red curve (hardly distinguished from the dashed one) is the same as as on the Figure 5 and represents the fit of data on structure function $F_2(x,Q^2)$ at the LO, the renormalon contributions of higher-twist terms included. The corresponding $\chi^2/\text{n.d.f.} = 555/660 = 0.84$.

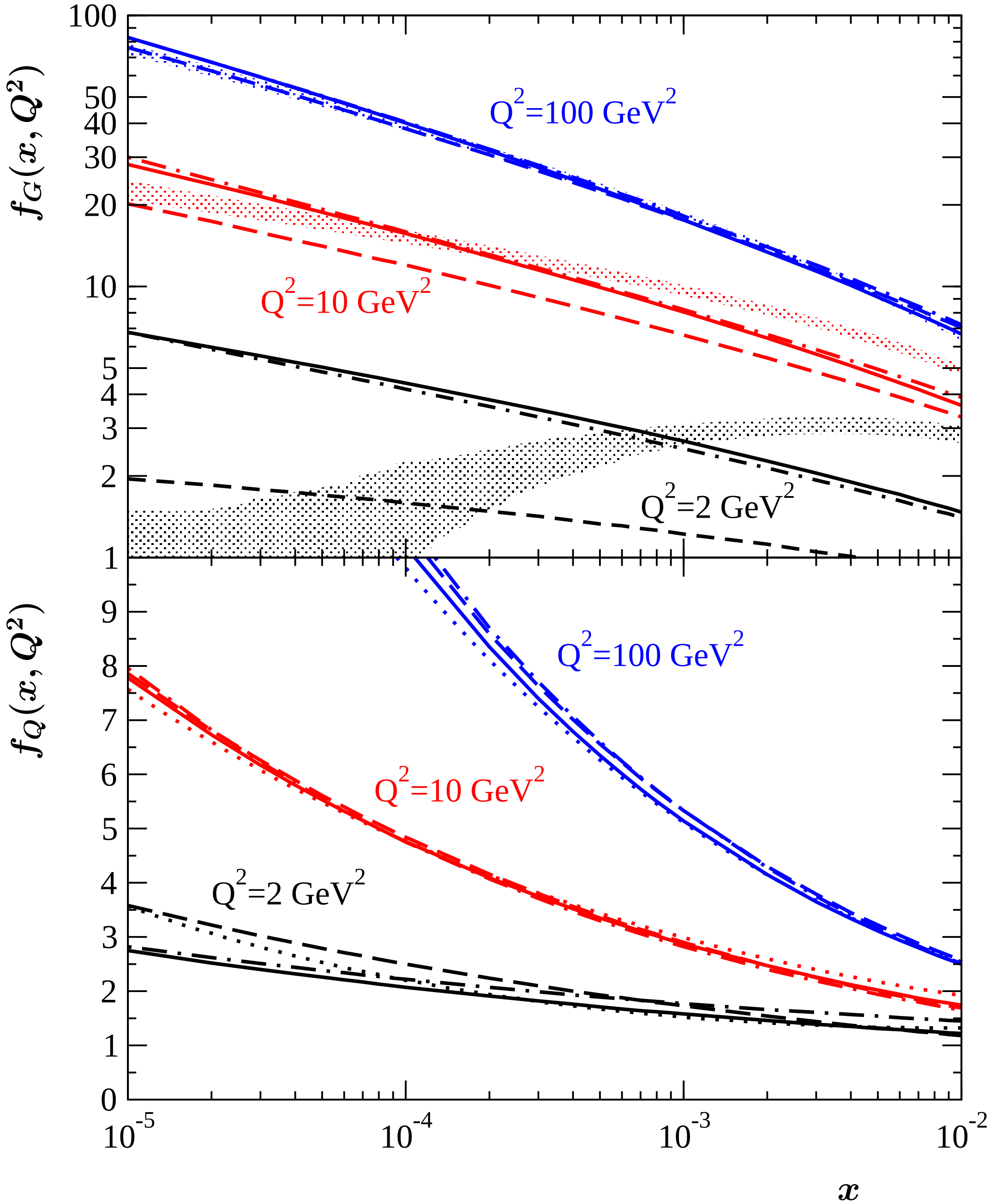

FIG. 9: The parton distributions $f_a(x,Q^2)$ as a function of x for $Q^2 = 2$, 10 and 100 GeV^2 compared to the NLO QCD predictions of A02NLO [88], represented by black dots. The solid lines represent the NLO fit alone with χ^2/n.d.f. = 798/611 = 1.31 [$A_G^{\tau 2} = -.020$, $A_q^{\tau 2} = .903$, $Q_0^2 = .495$ GeV^2]. The dash-dotted curves represent the BFKL-motivated estimation for the higher twist contribution with the value of the parameter $b = a^2/2$. The corresponding χ^2/n.d.f. = 629/609 = 1.03 [$A_G^{\tau 2} = .301$, $A_q^{\tau 2} = 0.535$, $Q_0^2 = .631$ GeV^2 and $A_G^{\tau 4} = -.580$ GeV^2, $A_q^{\tau 4} = 1.311$ GeV^2]. The dashed curves are obtained from the fits at the NLO, when the renormalon contributions of higher-twist terms have been incorporated. The corresponding χ^2/n.d.f. = 565/660 = 0.86 [$A_G^{\tau 2} = .279$, $A_q^{\tau 2} = .640$, $Q_0^2 = .672$ GeV^2 and $\mathrm{a}_G^{\tau 4} = -.143$ GeV^2, $\mathrm{a}_q^{\tau 4} = .140$ GeV^2, $\mathrm{a}_G^{\tau 6} = -.044$ GeV^4, $\mathrm{a}_q^{\tau 6} = .043$ GeV^4].